Unitary matrix integrals and enumerations of permutations

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Abstract.
We study an $N^{-1}$-expansion of a matrix integral over the unitary group $U(N)$. 

Let \( U(N) \) be the unitary group of degree \( N \):

\[
U(N) = \{ U = (u_{ij})_{1 \leq i,j \leq N} \in \text{GL}(N, \mathbb{C}) \mid UU^* = I \}.
\]

There exists the unique probability measure \( dU \) on \( U(N) \) satisfying

\[
d(V_1 U V_2) = dU \quad \text{for fixed } V_1, V_2 \in U(N); \\
\int_{U(N)} dU = 1.
\]

We call \( dU \) the normalized Haar measure on \( U(N) \).

We would like to evaluate the integral of the form \( \int_{U(N)} f(U) dU \), where \( f \) is a polynomial function in matrix entries \( u_{ij} \) and their complex conjugates \( \overline{u_{ij}} \).

Let \( S_n \) be the symmetric group acting on \( \{1, 2, \ldots, n\} \).
Number Theory

We show an importance of unitary matrix integrals. Keating and Snaith (2000) computed a unitary matrix integral:

\[
\int_{U(N)} |\det(I - U)|^{2m} dU = \prod_{j=1}^{N-1} \frac{j!(j + 2m)!}{((j + m)!)^2} \sim \left[ \prod_{j=0}^{m-1} \frac{j!}{(m + j)!} \right] N^{m^2}
\]

as \( N \to \infty \).

This is closely related to the moment of the Riemann zeta.

Keating-Snaith conjecture:

\[
\frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2m} dt \sim \left[ \prod_{j=0}^{m-1} \frac{j!}{(m + j)!} \right] a(m)(\log T)^{m^2},
\]

as \( T \to \infty \), where \( a(m) \) is a number-theoretic part.
General moments for a random unitary matrix

The Weingarten calculus developed since [Collins (2003)] gives a useful technique for the unitary matrix integral computation.

Given indices \( i_k, j_k, i'_k, j'_k \) \( (k = 1, 2, \ldots, n) \), we have

\[
\int_{U(N)} u_{i_1j_1} u_{i_2j_2} \cdots u_{i_nj_n} \frac{u_{i'_1j'_1} u_{i'_2j'_2} \cdots u_{i'_nj'_n}}{\prod_{k=1}^{n} \delta_{i_k, i'_{\sigma(k)}}} \prod_{k=1}^{n} \delta_{j_k, j'_{\tau(k)}} dU
\]

\[
= \sum_{\sigma, \tau \in S_n} \left( \prod_{k=1}^{n} \delta_{i_k, i'_{\sigma(k)}} \right) \left( \prod_{k=1}^{n} \delta_{j_k, j'_{\tau(k)}} \right) W_{g_n}^{U(N)}(\sigma^{-1} \tau).
\]

Here the unitary Weingarten function \( W_{g_n}^{U(N)}(\sigma) \) is defined by

\[
W_{g_n}^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N + j - i)} \chi^\lambda(\sigma) \quad (\sigma \in S_n).
\]
Representation theory

A weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l), \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$$

is said to be a partition of $n$ if $|\lambda| := \sum_{i=1}^{l} \lambda_i = n$. Then we write $\lambda \vdash n$. We call $\ell(\lambda) := l$ the length of $\lambda$.

Irreducible representations of $S_n$ are parametrized by partitions of $n$. $\chi^\lambda$: the irreducible character corresponding to $\lambda$. $f^\lambda$: the dimension of the irrep corresponding to $\lambda$.

**Example.** Let $n = 2$ and $\sigma = (1 \ 2)$. Then

$$Wg_2^{U(N)}((1 \ 2)) = \frac{1}{2!} \left( \frac{1}{N(N + 1)} \cdot 1 + \frac{1}{N(N - 1)} \cdot (-1) \right) = \frac{-1}{N(N^2 - 1)}.$$
Main Result (1)

We evaluate coefficients in the asymptotic expansion of $W_{g_n}^{U(N)}(\sigma)$ in the limit $N \to \infty$.

**Theorem [M-Novak (2009 preprint)]**

Suppose $N \geq n$. For each $\sigma \in S_n$ of cycle type $\mu$, we have

$$W_{g_n}^{U(N)}(\sigma) = (-1)^{n-\ell(\mu)} \sum_{k=0}^{\infty} \frac{a_{n-\ell(\mu)+2k}(\sigma)}{N^{2n-\ell(\mu)+2k}}$$

with

$$a_{n-\ell(\mu)}(\sigma) = \prod_{i=1}^{\ell(\mu)} \text{Cat}(\mu_i - 1), \quad \text{Cat}(m) = \frac{(2m)!}{(m+1)!m!}$$

and non-negative integers $a_{n-\ell(\mu)+2k}(\sigma)$. Furthermore...
Furthermore, the non-negative integer $a_p(\sigma)$ ($p \geq 0$, $\sigma \in S_n$) coincides with the number of sequences 
$(s_1, t_1, s_2, t_2, \ldots, s_p, t_p) \in \{1, 2, \ldots, n\}^{\times 2p}$ satisfying

- $s_i < t_i$ for all $i$.
- $1 < t_1 \leq t_2 \leq \cdots \leq t_p \leq n$.
- $\sigma = (s_1 \ t_1)(s_2 \ t_2) \cdots (s_p \ t_p)$.

**Example.** The value $a_2(id_n)$ coincides with the number of sequences $(s_1, t_1, s_2, t_2)$ satisfying

$$s_1 < t_1, \quad s_2 < t_2, \quad 2 \leq t_1 \leq t_2 \leq n, \quad id_n = (s_1 \ t_1)(s_2 \ t_2).$$

We have $s_1 = s_2$ and $t_1 = t_2$. Hence $a_2(id_n)$ equals the number of transpositions $(s_1 \ t_1)$ in $S_n$, i.e., $a_2(id_n) = \binom{n}{2}$.
Main Example

We obtain the expressions

\[
W_g^{U(N)}(\text{id}_n) = \int_{U(N)} |u_{11} u_{22} \cdots u_{nn}|^2 \, dU
\]

\[
= \frac{1}{n!} \sum_{\lambda \vdash n} \frac{(f^\lambda)^2}{\prod_{i=1}^\ell(\lambda) \prod_{j=1}^{\lambda_i} (N + j - i)}
\]

\[
= \sum_{k=0}^\infty \frac{a_{2k}(\text{id}_n)}{N^{n+2k}}
\]

\[
= 1N^{-n} + \binom{n}{2} N^{-n-2} + \left[ 3 \binom{n}{4} + 8 \binom{n}{3} + \binom{n}{2} \right] N^{-n-4} + O(N^{-n-6}).
\]

Remark. Our main result is closely related to Jucys-Murphy elements \( J_k = (1\ 2) + (2\ 3) + \cdots + (k-1\ k) \), which belong to the group algebra of \( S_n \).
Closing Remarks

The Weingarten calculus on the orthogonal group $O(N)$ has been developed in [Collins-Śniady (2006)] and [Collins-M (2009)]. The connection to an enumeration of permutations was given in [M (2010 preprint)]. However we need pairings (perfect matchings) instead of permutations.

Reference


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