Unitary matrix integrals and enumerations of permutations

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1. Unitary matrix integrals

2. Weingarten calculus on $U(N)$

3. Asymptotic behavior for Weingarten functions (Main Result)

4. Orthogonal matrix case
The unitary group:

\[ U(N) = \{ U = (u_{ij})_{1 \leq i,j \leq N} \in GL(N, \mathbb{C}) \mid UU^* = I \}. \]

Normalized Haar measure \( dU \):
there exists the unique probability measure \( dU \) on \( U(N) \) satisfying

\[ d(V_1 UV_2) = dU \quad \text{for fixed } V_1, V_2 \in U(N); \]

\[ \int_{U(N)} 1 dU = 1. \]
General problem

Given an integrable function $f$ on $U(N)$, we would like to compute the value of

$$\int_{U(N)} f(U) dU.$$ 

Furthermore, given a sequence $\{f_N\}$ of integrable functions $f_N$ on $U(N)$, we would like to evaluate the asymptotics of

$$\int_{U(N)} f_N(U) dU = a_0 N^p + a_1 N^{p-1} + a_2 N^{p-2} + \cdots \quad (N \to \infty)$$

with some constants $a_0, a_1, a_2, \cdots \in \mathbb{C}$ and $p \in \mathbb{Z}$.

**Example.** Consider $f_N(U) := |u_{11} u_{22}|^2$ for $U = (u_{ij})_{1 \leq i,j \leq N}$. Then we will have

$$\int_{U(N)} |u_{11} u_{22}|^2 dU = \frac{1}{N^2 - 1} = \frac{1}{N^2} + \frac{1}{N^4} + \frac{1}{N^6} + \cdots \sim N^{-2}.$$
Class functions

**Definition.**

A function $f$ on $U(N)$ is said to be a **class function** if it satisfies

$$f(V^{-1}UV) = f(V^* UV) = f(U) \quad (\text{for all } U, V \in U(N)).$$

Any unitary matrix can be diagonalized by a unitary matrix. Hence $f$ is a class function if and only if $f(U)$ depends on eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N}$ of $U$, i.e., $f$ satisfies

$$f(U) = f(\text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N})).$$
Weyl’s integration formula

If $f$ is an integrable class function, we can compute $\int_{U(N)} f(U) dU$ by the following formula.

**Theorem (Weyl)**

For any integrable class function $f$ on $U(N)$, we have

$$\int_{U(N)} f(U) dU = \frac{1}{(2\pi)^N N!} \int_{[0, 2\pi]^N} f(\text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N}))$$

$$\times \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 d\theta_2 \cdots d\theta_N.$$
Moments of the characteristic polynomial

Consider the integrable class function \( f(U) = |\det(I - U)|^{2m} \) with \( m \in \mathbb{N} \).

**Theorem [Keating-Snaith (2000)]**

\[
\int_{U(N)} |\det(I - U)|^{2m} \, dU = \prod_{j=1}^{N-1} \frac{j!(j + 2m)!}{((j + m)!)^2}.
\]

Hence we obtain its asymptotic behavior

\[
\int_{U(N)} |\det(I - U)|^{2m} \, dU \sim \left[ \prod_{j=0}^{m-1} \frac{j!}{(m+j)!} \right] N^{m^2} \quad (N \to \infty).
\]

Note: This quantity is related to moments of the Riemann zeta-function: \( \int_0^T |\zeta(\frac{1}{2} + t\sqrt{-1})|^{2m} \, dt \) as \( T \to \infty \).
A problem

How can we compute the integral \( \int_{U(N)} f(U) \, dU \) unless \( f \) is a class function?

In general, we would like to compute an integral for a polynomial function of the form

\[
f(U) = \left( \sum_{i,j=1}^{N} a_{ij} u_{ij}^{m_{ij}} \right) \left( \sum_{i,j=1}^{N} b_{ij} \overline{u_{ij}}^{n_{ij}} \right)
\]

with \( a_{ij}, b_{ij} \in \mathbb{C} \) and \( m_{ij}, n_{ij} \in \mathbb{N} \). To do it, we will observe integrals

\[
\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1}} u_{i'_2 j'_2} \cdots u_{i'_m j'_m} \, dU
\]

where \( i_k, j_k, i'_k, j'_k \) are indices from \( \{1, 2, \ldots, N\} \).
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Weingarten calculus

The calculus for the integral

\[ \int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i_1' j_1'}} u_{i_2' j_2'} \cdots u_{i_m' j_m'} dU \]

was first studied by D. Weingarten (1978), and developed by B. Collins with his coauthors since 2003. The calculus is called the **Weingarten calculus**.

**Claim.**

\[ \int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i_1' j_1'}} u_{i_2' j_2'} \cdots u_{i_m' j_m'} dU = 0 \]

unless \( n = m \).
Let $S_n$ be the symmetric group acting on $\{1, 2, \ldots, n\}$.

**Theorem [Collins (2003)]**

\[
\int_{U(N)} u_{i_1j_1} u_{i_2j_2} \cdots u_{i_nj_n} u_{i'_1j'_1} u_{i'_2j'_2} \cdots u_{i'_nj'_n} dU
\]

\[
= \sum_{\sigma, \tau \in S_n} \left( \prod_{k=1}^{n} \delta_{i_k, i'_\sigma(k)} \right) \left( \prod_{k=1}^{n} \delta_{j_k, j'_\tau(k)} \right) W_{g_n}^{U(N)}(\sigma^{-1}\tau).
\]

Here the function $W_{g_n}^{U(N)}(\sigma)$ is given by

\[
W_{g_n}^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N + j - i)} \chi^\lambda(\sigma) \quad (\sigma \in S_n),
\]

which is called the **unitary Weingarten function**.
Representation theory

A weakly decreasing sequence of positive integers

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l), \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0) \]

is said to be a partition of \( n \) if \( |\lambda| := \sum_{i=1}^{l} \lambda_i = n \). Then we write \( \lambda \vdash n \). We call \( \ell(\lambda) := l \) the length of \( \lambda \).

Irreducible representations of \( S_n \) are parametrized by partitions of \( n \).
\( \chi^\lambda \): the irreducible character corresponding to \( \lambda \).
\( f^\lambda \): the dimension of the irrep corresponding to \( \lambda \).

**Example.** Let \( n = 2 \) and \( \sigma = (1 \ 2) \). Then

\[ Wg_2^{U(N)}((1 \ 2)) = \frac{1}{2!} \left( \frac{1}{N(N+1)} \cdot 1 + \frac{1}{N(N-1)} \cdot (-1) \right) = \frac{-1}{N(N^2 - 1)}. \]
Example

\[
\int_{U(N)} u_{11} u_{12}^2 u_{12} u_{22} dU = 0.
\]

\[
\int_{U(N)} u_{11} u_{12} u_{12} u_{11} u_{12} u_{22} dU = 0.
\]

(The row indices are (1, 1, 1) and (1, 1, 2)).

\[
\int_{U(N)} u_{11} u_{12} u_{12} u_{11} u_{12} u_{12} dU = \sum_{\sigma \in S_3} \sum_{\tau \in \{\text{id}_3, (2 3)\}} W_{g_3}^{U(N)}(\sigma^{-1} \tau)
\]

\[
= 2 \sum_{\sigma \in S_3} W_{g_3}^{U(N)}(\sigma) = \frac{2}{N(N + 1)(N + 2)}.
\]

row indices: (1, 1, 1) and (1, 1, 1).

column indices: (1, 2, 2) and (1, 2, 2).
Example

In general, if \( N \geq n \),

\[
W_{g_n}^{U(N)}(\sigma) = \int_{U(N)} \prod_{i=1}^{n} u_{ii} u_{i\sigma(i)} \, dU \quad (\sigma \in S_n).
\]

Example.

\[
\int_{U(N)} u_{11} u_{22} u_{33} u_{12} u_{23} u_{31} \, dU = W_{g_3}^{U(3)}((1 \ 2 \ 3)) = \frac{2}{(N+2)(N+1)N(N-1)(N-2)}.
\]
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Let $\mu = (\mu_1, \mu_2, \ldots)$ be a partition of $n$.

A permutation $\sigma \in S_n$ is said to be of cycle-type $\mu$ if the cycle decomposition of $\sigma$ is of the form

$$\sigma = (i_1 i_2 \cdots i_{\mu_1})(j_1 j_2 \cdots j_{\mu_2}) \cdots.$$

**Example.** In $S_6$,

$$\sigma = (1 \ 3 \ 2)(4 \ 6)(5) \quad \mu = (3, 2, 1)$$

$$\quad (2 \ 3)(5 \ 6)(1)(4) \quad (2, 2, 1, 1)$$

$$\quad (1 \ 3 \ 5 \ 6 \ 2 \ 4) \quad (6)$$

$$\text{id}_6 = (1)(2)(3)(4)(5)(6) \quad (1, 1, 1, 1, 1, 1)$$
Asymptotic behavior

**Theorem [M-Novak (2009 preprint)]**

Suppose $N \geq n$. For each $\sigma \in S_n$ of cycle type $\mu$, we have

$$Wg_n^{U(N)}(\sigma) = (-1)^{n-\ell(\mu)} \sum_{k=0}^{\infty} \frac{a_{n-\ell(\mu)+2k}(\sigma)}{N^{2n-\ell(\mu)+2k}}$$

with

$$a_{n-\ell(\mu)}(\sigma) = \prod_{i=1}^{\ell(\mu)} \text{Cat}(\mu_i - 1), \quad \text{Cat}(m) = \frac{(2m)!}{(m+1)!m!}$$

and non-negative integers $a_{n-\ell(\mu)+2k}(\sigma)$. Furthermore...
Asymptotic behavior

Furthermore, the non-negative integer $a_p(\sigma) \quad (p \geq 0, \quad \sigma \in S_n)$ coincides with the number of sequences $(s_1, t_1, s_2, t_2, \ldots, s_p, t_p) \in \{1, 2, \ldots, n\}^{\times 2p}$ satisfying

- $s_i < t_i$ for all $i$.
- $1 < t_1 \leq t_2 \leq \cdots \leq t_p \leq n$.
- $\sigma = (s_1 \ t_1)(s_2 \ t_2) \cdots (s_p \ t_p)$.

**Example.** The value $a_2(\text{id}_n)$ coincides with the number of sequences $(s_1, t_1, s_2, t_2)$ satisfying

$$s_1 < t_1, \quad s_2 < t_2, \quad 2 \leq t_1 \leq t_2 \leq n, \quad \text{id}_n = (s_1 \ t_1)(s_2 \ t_2).$$

Then we have $s_1 = s_2$ and $t_1 = t_2$. Hence $a_2(\text{id}_n)$ equals the number of transpositions $(s_1 \ t_1)$ in $S_n$, i.e., $a_2(\text{id}_n) = \binom{n}{2}$. 
Consider the case of $\sigma = \text{id}_n$.

$$W_{g_n}^{U(N)}(\text{id}_n) = \int_{U(N)} |u_{11} u_{22} \cdots u_{nn}|^2 dU$$

$$= \frac{1}{n!} \sum_{\lambda \vdash n} \frac{(f^\lambda)^2}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N + j - i)}$$

$$= \sum_{k=0}^{\infty} \frac{a_{2k}(\text{id}_n)}{N^{n+2k}}$$

$$= 1N^{-n} + \binom{n}{2} N^{-n-2} + \left[ 3 \binom{n}{4} + 8 \binom{n}{3} + \binom{n}{2} \right] N^{-n-4} + O(N^{-n-6}).$$
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Orthogonal matrix integrals

Consider the orthogonal group

\[ O(N) = \{ V = (v_{ij})_{1 \leq i,j \leq N} \in GL(N, \mathbb{R}) \mid ^tVV = I \} . \]

The group also has the normalized Haar measure \( dV \).

Consider the integral

\[ \int_{O(N)} f(V) dV \]

where \( f \) is a polynomial function in variables \( v_{ij} \). The discussion is parallel to the unitary case but a bit more complicated.

The Weingarten calculus on \( O(N) \) has been developed in [Collins-Śniady (2006)] and [Collins-M (2009)].

The connection to an enumeration of permutations was given in [M (2010 preprint)]. However we need pairings (perfect matchings) instead of permutations.