On Moments of entries of a COE matrix

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1. Introduction

2. COE

3. Orthogonal Weingarten Function

4. Applications of Main Theorem

5. Conclusion
Consider a random matrix

\[ X = (x_{ij})_{1 \leq i, j \leq N}. \]

(ex. Gaussian matrix, Wishart matrix, Haar-distributed unitary matrix, etc.)

**Problem**

How can we compute the following mixed moments?

\[ \mathbb{E}[x_{i_1j_1} x_{i_2j_2} \cdots x_{i_nj_n}] \]

or

\[ \mathbb{E}[x_{i_1j_1} x_{i_2j_2} \cdots x_{i_nj_n} x_{k_1l_1} x_{k_2l_2} \cdots x_{k_nl_n}] \]
**Gaussian matrix**: well-known Wick formula. If $y_1, y_2, y_3, y_4$ are Gaussian r.v., then

\[
\mathbb{E}[y_1 y_2 y_3 y_4] = \mathbb{E}[y_1 y_2] \mathbb{E}[y_3 y_4] \\
+ \mathbb{E}[y_1 y_3] \mathbb{E}[y_2 y_4] + \mathbb{E}[y_1 y_4] \mathbb{E}[y_2 y_3].
\]

**central complex Wishart matrix and its inverse matrix**: [Graczyk-Letac-Massam 03].

**central real Wishart matrix and its inverse matrix**: [Graczyk-Letac-Massam 05], [M 11].

**noncentral Wishart matrix**: [Kuriki-Numata 10].
History

- **Haar-distributed unitary matrix**: [Samuel 80], [Weingarten 78], [Collins 03]. We call their technique *Weingarten calculus*.

- **Haar-distributed orthogonal matrix**: [Collins-Śniady 06], [Collins-M 09].

- **Dyson’s circular ensembles:**
  - circular unitary ensemble (CUE) = Unitary group with Haar measure.
  - circular orthogonal ensemble (COE) — *Today’s topic*.
  - circular symplectic ensemble (CSE) — in future.
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Definition of COE

\[ \text{COE}(N) := \{ N \times N \text{ symmetric unitary matrices} \}. \]

Fact and Definition

There is a unique random matrix \( V \in \text{COE}(N) \) such that

\[ V \text{ and } W_0^T VW_0 \text{ have the same distribution} \]

where \( W_0 \) is an \( N \times N \) fixed unitary matrix.

We call this \( V \) a COE matrix.

Recall that a CUE matrix (or a Haar-distributed unitary matrix) \( U \) is a random matrix in the unitary group \( U(N) \) such that

\[ U \text{ and } W_1 UW_2 \text{ have the same distribution} \]

where \( W_1, W_2 \) are \( N \times N \) fixed unitary matrices.
Our Problem

Let $V = (v_{ij})_{1 \leq i, j \leq N}$ be a COE matrix. Let $i = (i_1, i_2, \ldots, i_{2n})$ and $j = (j_1, j_2, \ldots, j_{2n})$ be two sequences in $\{1, 2, \ldots, N\} \times 2^n$.

Let

$$M_N(i, j) := \mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2n-1} i_{2n}} \overline{v_{j_1 j_2}} \overline{v_{j_3 j_4}} \cdots \overline{v_{j_{2n-1} j_{2n}}}].$$

Example.

$$M_N((1234), (1324)) = \mathbb{E}[v_{12} v_{34} \overline{v_{13}} \overline{v_{24}}].$$
$$M_N((1212), (1212)) = \mathbb{E}[v_{12} v_{12} \overline{v_{12}} \overline{v_{12}}] = \mathbb{E}[|v_{12}|^4].$$

Problem

Give a method for computing the moments $M_N(i, j)$. 
Main Theorem

Let $M_N(i, j)$ be as above. Then we have

$$M_N(i, j) = \sum_{\sigma \in S_{2n}} W_{g_n}^{O(N+1)}(\sigma),$$

where the sum runs over permutations $\sigma$ in the symmetric group $S_{2n}$ satisfying

$$j = i^\sigma := (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(2n)}),$$

and $W_{g_n}^{O(N+1)}$ is the orthogonal Weingarten function. (We will give the definition of the orthogonal Weingarten function later.)
How to use Main Theorem

Example.

\[ \mathbb{E}[\nu_{12} \nu_{34} \overline{\nu_{13}} \nu_{24}] = M_N((1234), (1324)) \]
\[ = W g_2^{O(N+1)} \left( \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \right) = \frac{-1}{N(N+1)(N+3)}. \]

Example.

\[ \mathbb{E}[|\nu_{11}|^2] = \mathbb{E}[\nu_{11} \nu_{11} \overline{\nu_{11}} \nu_{11}] = M_N((1111), (1111)) \]
\[ = \sum_{\sigma \in S_4} W g_2^{O(N+1)}(\sigma) = \frac{8}{(N+1)(N+3)}. \]

Example. Since \( j = (1112) \) is not a rearrangement of \( i = (1212) \),

\[ \mathbb{E}[\nu_{12}^2 \nu_{11} \nu_{12}] = M_N((1212), (1112)) = 0. \]
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Definition of Orthogonal Wg

The hyperoctahedral group $H_n$ is the subgroup of $S_{2n}$ generated by

$$(2k - 1 \ 2k), \quad 1 \leq k \leq n,$$

$$(2i - 1 \ 2j - 1)(2i \ 2j), \quad 1 \leq i < j \leq n.$$ 

Well-known Fact

Double cosets

$$\{ H_n\sigma H_n \mid \sigma \in S_{2n} \}$$

are parametrized by partitions of $n$. Hence, we have the decomposition

$$S_{2n} = \bigsqcup_{\mu \vdash n} H_\mu,$$

where each $H_\mu$ is of the form $H_n \sigma H_n$ for some $\sigma \in S_{2n}$. 
Definition of Orthogonal Wg

Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition of \( n \).

\[
C'_\lambda(N) := \prod_{(i,j) \in \lambda} (N + 2j - i - 1),
\]

where the product runs over all boxes of the Young diagram of \( \lambda \).

The zonal spherical function for the pair \((S_{2n}, H_n)\) is defined by

\[
\omega^\lambda(\sigma) := (2^n n!)^{-1} \sum_{\tau \in H_n} \chi^{2\lambda}(\sigma \tau), \quad (\sigma \in S_{2n})
\]

where \( \chi^{2\lambda} \) is the irreducible character of \( S_{2n} \) associated with \( 2\lambda = (2\lambda_1, 2\lambda_2, \ldots) \vdash 2n \).

\[
f^{2\lambda} := \chi^{2\lambda}(\text{id}_{S_{2n}}) = \# \text{ of standard Young tableaux of shape } 2\lambda.
\]
Definition of Orthogonal $W_g$

**Definition**

The **orthogonal Weingarten function** is defined by

$$W_g^{O(N)}(\sigma) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C'_\lambda(N)} \omega^\lambda(\sigma) \quad (\sigma \in S_{2n}).$$

Example ($n = 2$). If $\sigma = (2 \ 3) \in S_4$,

$$W_{g_2}^{O(N)}(\sigma) = \frac{1}{3} \left( \frac{1 \cdot 1}{N(N+2)} + \frac{2 \cdot (-1/2)}{N(N-1)} \right) = \frac{-1}{N(N+2)(N-1)}.$$
Example

Fact

$W_{n}^{O(N)}$ takes constant at each double coset $H_{\mu}$.

If $n = 3$, then $S_{6} = H_{(3)} \sqcup H_{(2,1)} \sqcup H_{(1,1,1)}$.

$W_{3}^{O(N)}(\sigma)$ are given by

$$
\frac{2}{N(N + 2)(N + 4)(N - 1)(N - 2)} - \frac{1}{N(N + 4)(N - 1)(N - 2)} + \frac{N^2 + 3N - 2}{N(N + 2)(N + 4)(N - 1)(N - 2)}
$$

$(\sigma \in H_{(3)})$,

$(\sigma \in H_{(2,1)})$,

$(\sigma \in H_{(1,1,1)})$. 
Main Theorem (again)

Let $V = (v_{ij})_{1 \leq i, j \leq N}$ be a COE matrix. Let $i = (i_1, i_2, \ldots, i_{2n})$ and $j = (j_1, j_2, \ldots, j_{2n})$ be two sequences in $\{1, 2, \ldots, N\} \times 2^n$, and let

$$M_N(i, j) := \mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2n-1} i_{2n}} v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2n-1} j_{2n}}].$$

Then we have

$$M_N(i, j) = \sum_{\sigma \in S_{2n}} W g_n^{O(N+1)}(\sigma).$$
Haar-distributed orthogonal matrix

Compare Main Theorem with the following theorem.

**Theorem. [Collins-Śniady (06), Collins-M (09)]**

Let \( X = (x_{ij})_{1 \leq i,j \leq N} \) be a Haar-distributed orthogonal matrix taken from \( O(N) \). For two sequences \( \mathbf{i} = (i_1, \ldots, i_{2n}) \) and \( \mathbf{j} = (j_1, \ldots, j_{2n}) \),

\[
\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_{2n} j_{2n}}] = (2^n n!)^{-2} \sum_{\sigma} \sum_{\tau} W_{g_n}^{O(N)}(\sigma^{-1} \tau),
\]

where the sum runs over all \( \sigma, \tau \in S_{2n} \) satisfying

\[
i_{\sigma}(2k-1) = i_{\sigma}(2k), \quad j_{\tau}(2k-1) = j_{\tau}(2k) \quad \text{for all} \ 1 \leq k \leq n.
\]
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**Corollary 1**

Let $v_{ii}$ be a diagonal entry of an $N \times N$ COE matrix. Then, for $n \geq 1$,

$$
\mathbb{E}[|v_{ii}|^{2n}] = \frac{2^n n!}{(N + 1)(N + 3) \cdots (N + 2n - 1)}.
$$

**Proof.**

$$
\mathbb{E}[|v_{ii}|^{2n}] = M_N((i, \ldots, i), (i, \ldots, i))
\begin{align*}
&= \sum_{\sigma \in S_{2n}} W_{2n}^{O(N+1)}(\sigma) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C'_\lambda(N + 1)} \sum_{\sigma \in S_{2n}} \omega^\lambda(\sigma) \\
&= 2^n n! \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C'_\lambda(N + 1)} \delta_{\lambda, (n)} = \frac{2^n n!}{(N + 1)(N + 3) \cdots (N + 2n - 1)}.
\end{align*}
$$
Applications

**Corollary 2**

Let $v_{ij}$ be an off-diagonal entry of an $N \times N$ COE matrix. Then, for $n \geq 1$,

$$
\mathbb{E}[|v_{ij}|^{2n}] = \frac{n!}{N(N + 1)(N + 2) \cdots (N + n - 2) \cdot (N + 2n - 1)}.
$$

**Proof.**

\[
\mathbb{E}[|v_{ij}|^{2n}] = M_N((i, j, i, j, \ldots, i, j), (i, j, i, j, \ldots, i, j)) \\
= \sum_{\sigma} W_{g_n}^{O(N+1)}(\sigma),
\]

summed over all $\sigma \in S_{2n}$, each of which permutates odd numbers and even numbers. This case is more difficult than Corollary 1...
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Conclusion

We have studied the following problem.

**Problem**

Given a random matrix $X = (x_{ij})_{1 \leq i, j \leq N}$, how can we compute the following mixed moments?

$$
\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n}] \quad \text{or} \quad \mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} x_{k_1 l_1} x_{k_2 l_2} \cdots x_{k_n l_n}]
$$

This problem has been completely solved for:
- Gaussian matrix, Wishart matrix, unitary group $U(N)$, orthogonal group $O(N)$...
- Dyson’s circular orthogonal ensemble $\text{COE}(N)$ New!

The orthogonal Weingarten function $W_{g_n}^{O(N+1)}$ is a key item.