Jucys-Murphy elements and matrix integrals – recent developments

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Related papers and people

- Alain Lascoux and Jean-Yves Thibon (2001), *Vertex operators and the class algebras of symmetric groups.*
- Benoît Collins and Sho Matsumoto (2009), *On some properties of orthogonal Weingarten functions* .
- Sho Matsumoto and Jonathan Novak, (2009 preprint), *Jucys-Murphy elements and unitary matrix integrals.*
- Sho Matsumoto (2010 preprint), *Jucys-Murphy elements, orthogonal matrix integrals, and Jack measures.*
- Michel Lassalle (2010 preprint), *Class expansion of some symmetric functions in Jucys-Murphy elements* .
- Valentin Féray (2010 preprint), *New induction relations for homogeneous functions in Jucys-Murphy elements.*

Jucys-Murphy elements

Let S_n be the symmetric group.
Jucys-Murphy elements are elements in the group algebra $\mathbb{C}[S_n]$.
They are defined by

$$J_1 = 0.$$
$$J_2 = (1\ 2).$$
$$J_3 = (1\ 3) + (2\ 3).$$
$$J_4 = (1\ 4) + (2\ 4) + (3\ 4).$$
$$\ldots$$
$$J_n = (1\ n) + (2\ n) + (3\ n) + \cdots + (n-1\ n).$$

They are commutative: $J_k J_l = J_l J_k$.

Properties of JM-elements

Let λ be a partition of n, and let $(\rho_\lambda, V_\lambda)$ be the irreducible representation of S_n associated with λ.
Then there exists a basis

$$\{e_T \mid T : \text{standard Young tableaux of shape } \lambda\}$$

of V_λ such that

$$\rho_\lambda(J_k)e_T = c_T(k)e_T \qquad (k = 1, 2, \ldots, n).$$

Here $c_T(k)$ is the content $j - i$ of the box labelled k in T.

Properties of JM-elements

Example. Let

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}.$$

The list of contents is

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & 0 \\ \cline{1-2}\end{array} \qquad \text{(the content of the } (i,j)\text{-th box is } j-i\text{).}$$

$$J_1 e_T = 0 e_T, \qquad J_2 e_T = (-1) e_T, \qquad J_3 e_T = 1 e_T,$$
$$J_4 e_T = 2 e_T, \qquad J_5 e_T = 0 e_T.$$

Here we write as $J_k e_T := \rho_\lambda(J_k) e_T$.

Jucys's and Murphy's theorems

Let Λ be the algebra of symmetric functions in infinitely-many variables in complex coeffcients.

Theorem [Jucys (1974)]

For each $F \in \Lambda$, the element $F(J_1, J_2, \dots, J_n)$ in $\mathbb{C}[S_n]$ belongs to the center $Z(\mathbb{C}[S_n])$.

Conversely,

Theorem [Murphy (1981)]

$$Z(\mathbb{C}[S_n]) = \{F(J_1, J_2, \dots, J_n) \mid F \in \Lambda\}.$$

Our problems

The class algebra $Z(\mathbb{C}[S_n])$ has two bases

$$\{\chi^\lambda \mid \lambda \vdash n\} \qquad \text{(irreducible characters)},$$
$$\{C_\mu \mid \mu \vdash n\} \qquad \text{(class sums)}.$$

Here C_μ is the sum of all permutations in S_n of cycle-types μ.

A natural question

Given a symmetric fuction F (e.g. $F = e_k, h_k, p_k, m_\lambda$), can we obtain an explicit expansion of $F(J_1, J_2, \ldots, J_n)$ in χ^λ (resp. C_μ)?

An answer

Theorem [Jucys (1974)]

For each symmetric function F,

$$F(J_1, J_2, \dots, J_n) = \sum_{\lambda \vdash n} F(A_\lambda) \frac{f^\lambda}{n!} \chi^\lambda.$$

Here A_λ is the alphabet of all contents of λ and f^λ is the number of standard Young tableaux: $f^\lambda = \dim V_\lambda$.

Example.

$$F(J_1, J_2, J_3) = F(0, 1, 2)\frac{1}{6}\chi^{(3)} + F(0, 1, -1)\frac{2}{6}\chi^{(2,1)} + F(0, -1, -2)\frac{1}{6}\chi^{(1^3)}.$$

Main problem in this talk

Definition.

Let F be a symmetric function. Define coefficients $a_\mu(F)$ by

$$F(J_1, J_2, \ldots, J_n) = \sum_{\lambda \vdash n} a_\mu(F) C_\mu.$$

Main problem

Evaluate $a_\mu(F)$.

Example

$$p_1(J_1, J_2, \dots, J_n) = J_1 + J_2 + \cdots + J_n = \sum_{1 \le i < j \le n} (i\ j) = C_{(2,1^{n-2})}.$$

In other words, $a_\mu(p_1) = 1$ if $\mu = (2, 1^{n-2})$ (i.e. $\ell(\mu) = n - 1$) and $a_\mu(p_1) = 0$ otherwise.

$$h_2(J_1, J_2, \dots, J_n) = \sum_{1 \le k \le l \le n} J_k J_l = \sum_{1 \le k \le l \le n} \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} (i\ k)(j\ l)$$
$$= 2C_{(3,1^{n-3})} + C_{(2^2,1^{n-4})} + \binom{n}{2} C_{(1^n)}.$$

Weingarten calculus

Symmetric functions in Jucys-Murphy elements are related to unitary matrix integrals.

Let $U(N)$ be the unitary group of degree N and let $\mathrm{d}U$ be its normalized Haar measure. Consider an integrable function f on $U(N)$. We would like to compute an integral

$$\int_{U \in U(N)} f(U) \mathrm{d}U.$$

We can apply Weyl's integration formula if f is central.

How can we compute an integral for non-central f ?

In general, we would like to compute an integral for a polynomial in matrix coordinates u_{ij} and their complex conjugates $\bar{u}_{ij}$.

Weingarten calculus

Theorem [Collins (2003)]

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_n j'_n}} \mathrm{d}U$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma \cdot (i_1,\dots,i_n) = (i'_1,\dots,i'_n)}} \sum_{\substack{\tau \in S_n \\ \tau \cdot (j_1,\dots,j_n) = (j'_1,\dots,j'_n)}} \mathrm{Wg}_n^{U(N)}(\sigma^{-1}\tau).$$

Here $\mathrm{Wg}_n^{U(N)}$ is the class function on S_n given by

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n : \ell(\lambda) \le N} \frac{f^\lambda}{\prod_{(i,j) \in \lambda} (N + j - i)} \chi^\lambda(\sigma) \qquad (\sigma \in S_n),$$

which is called the unitary Weingarten function.

Example of Weingarten calculus

$$\int_{U(N)} u_{11} u_{12} u_{12} \overline{u_{11} u_{12} u_{22}} \mathrm{d}U = 0.$$

$$\int_{U(N)} u_{11} u_{12} u_{12} \overline{u_{11} u_{12} u_{12}} \mathrm{d}U$$

$$= \sum_{\sigma \in S_3} \sum_{\tau \in \{\mathrm{id}_3, (2\ \ 3)\}} \mathrm{Wg}_3^{U(N)}(\sigma^{-1}\tau)$$

$$= 2 \sum_{\sigma \in S_3} \mathrm{Wg}_3^{U(N)}(\sigma) = \frac{2}{N(N+1)(N+2)}.$$

$$\int_{U(N)} u_{11} u_{22} u_{33} \overline{u_{12} u_{23} u_{31}} \mathrm{d}U$$

$$= \mathrm{Wg}_3^{U(N)}((1\ 2\ 3)) = \frac{2}{(N+2)(N+1)N(N-1)(N-2)}.$$

Weingarten and Jucys-Murphy

Let h_k be the k-th complete homogeneous symmetric polynomial

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{\substack{a_1, a_2, \ldots, a_n \geq 0 \\ a_1 + a_2 + \cdots + a_n = k}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Recall that coefficients $a_\mu(h_k)$ are defined by $h_k(J_1, \ldots, J_n) = \sum_{\mu \vdash n} a_\mu(h_k) C_\mu$.

Theorem [M-Novak (2009 preprint)]

Suppose $N \geq n$. For each $\sigma \in S_n$ of cycle-type $\mu \vdash n$,

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \sum_{k=0}^{\infty} (-1)^k a_\mu(h_k) N^{-n-k}.$$

Example

$$\mathrm{Wg}_n^{U(N)}(\mathrm{id}_n) = \int_{U(N)} |u_{11}u_{22}\cdots u_{nn}|^2 \mathrm{d}U$$

$$= \sum_{k=0}^{\infty} \frac{a_{(1^n)}(h_{2k})}{N^{n+2k}}$$

$$= N^{-n} + \binom{n}{2} N^{-n-2} + \left[3\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2}\right] N^{-n-4} + O(N^{-n-6}).$$

General properties

Let F be a homogeneous symmetric function of degree k.

$$F(J_1, J_2, \ldots, J_n) = \sum_{\mu \vdash n} a_\mu(F) C_\mu.$$

Proposition [M-Novak (2009 preprint)]

1. $a_\mu(F)$ is zero unless $n - \ell(\mu) \leq k$.
2. $a_\mu(F)$ is zero unless $n - \ell(\mu) \equiv k \pmod 2$.
3. If $n - \ell(\mu) = k$ then $a_\mu(F)$ is independent of n. (We call $\{a_\mu(F) \mid n - \ell(\mu) = k\}$ leading coefficients.)
4. In general, $a_\mu(F)$ are polynomials in n. (The case with $\mu = (1^n)$ was first proved by Stanley (2010)).

Leading coefficients

There exists a one-to-one correspondence between

$$\{\mu \vdash n \mid n - \ell(\mu) = k\} \quad \text{and} \quad \{\text{partitions of } k\}.$$

The map is

$$\mu = (\mu_1, \dots \mu_{\ell(\mu)}) \mapsto (\mu_1 - 1, \dots, \mu_{\ell(\mu)} - 1).$$

ex.

$$(4, 1^{n-4}) \mapsto (3)$$
$$(3, 2, 1^{n-5}) \mapsto (2, 1)$$
$$(2^3, 1^{n-6}) \mapsto (1^3)$$

Elementary sym fun e_k

Let

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Theorem [Jucys (1974)]

$$e_k(J_1, \ldots, J_n) = \sum_{\mu \vdash n : n - \ell(\mu) = k} C_\mu.$$

In other words, all leading coefficients are 1, and other coefficients $a_\mu(e_k)$ are zero.

Example

$$e_1(J_1, J_2, \dots, J_n) = C_{(2,1^{n-1})}.$$

$$e_2(J_1, J_2, \dots, J_n) = C_{(3,1^{n-3})} + C_{(2^2,1^{n-4})}.$$

$$e_3(J_1, J_2, \dots, J_n) = C_{(4,1^{n-4})} + C_{(3,2,1^{n-5})} + C_{(2^3,1^{n-6})}.$$

$$e_4(J_1, J_2, \dots, J_n) = C_{(5,1^{n-5})} + C_{(4,2,1^{n-6})} + C_{(3^2,1^{n-6})} + C_{(3,2^2,1^{n-7})} + C_{(2^4,1^{n-8})}.$$

Complete sym fun h_k

Consider

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Theorem.

The leading coefficients are given as follows: If $n - \ell(\mu) = k$, then

$$a_\mu(h_k) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}(\mu_i - 1), \qquad \mathrm{Cat}(m) = \frac{(2m)!}{(m+1)!m!}.$$

Algebraic proof [Murray (2004)] (in the framework of the Farahat-Higman algebra).
Combinatorial proof [M-Novak (2009 preprint)].

Complete sym fun h_k

We could not obtain any explicit expression for non-leading coefficients $a_\mu(h_k)$ $(n - \ell(\mu) \leq k-2)$.

Theorem [Lassalle (2010 preprint)], [Féray (2010 preprint)]

Coefficients $a_\mu^k := a_\mu(h_k)$ satisfy the following reccurance identity:

$$a^k_{\rho\cup(m)} = \delta_{m,1} a^k_\rho + \sum_{i=1}^{\ell(\rho)} \rho_i a^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{r=1}^{m-1} a^{k-1}_{\rho\cup(r)\cup(m-r)}.$$

Here

$\rho \setminus (\rho_i)$: the partition obtained by removing a part ρ_i from ρ,

$\rho \cup (m)$: the partition obtained by adding a part m to ρ.

Complete sym fun h_k

Theorem [Lassalle (2010 preprint)], [Féray (2010 preprint)]

Coefficients $a_\mu(h_k)$ are of the form

$$a_\mu(h_k) = \sum_{i=0}^{m_1(\mu)} c^k_{\overline{\mu}\cup(1^i)} \binom{m_1(\mu)}{i},$$

where $\overline{\mu}$ is obtained from μ by erasing its parts equal to 1 (i.e. $\mu = \overline{\mu} \cup (1^{m_1(\mu)})$). The c^k_ρ satisfy

$$c^k_{\rho\cup(m)} = \sum_{i=1}^{\ell(\rho)} \rho_i c^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{r=1}^{m-1} c^{k-1}_{\rho\cup(r)\cup(m-r)} + \delta_{m\geq 2} 2 c^{k-1}_{\rho\cup(m-1)} + \delta_{m,2} c^{k-1}_\rho.$$

Example

$$h_2(J_1, J_2, \dots, J_n) = 2\, C_{(3,1^{n-3})} + 1\, C_{(2^2,1^{n-4})} + \binom{n}{2} C_{(1^n)}.$$

$$h_3(J_1, J_2, \dots, J_n) = 5\, C_{(4,1^{n-4})} + 2\, C_{(3,2,1^{n-5})} + 1\, C_{(2^3,1^{n-6})} + \left[\binom{n-2}{2} + 4\binom{n-2}{1} + \binom{n-2}{0} \right] C_{(2,1^{n-2})}.$$

$$h_4(J_1, J_2, \dots, J_n) = 14\, C_{(5,1^{n-5})} + 5\, C_{(4,2,1^{n-6})} + 4\, C_{(3^2,1^{n-6})} + 2\, C_{(3,2^2,1^{n-7})} + 1\, C_{(2^4,1^{n-8})} + \left[2\binom{n-3}{2} + 15\binom{n-3}{1} + 10 \right] C_{(3,1^{n-3})} + \left[\binom{n-4}{2} + 8\binom{n-4}{1} + 20 \right] C_{(2^2,1^{n-4})} + \left[3\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2} \right] C_{(1^n)}.$$

Power-sum sym fun p_k

Consider

$$p_k(J_1, J_2, \ldots, J_n) = J_1^k + J_2^k + \cdots + J_n^k.$$

It is easy to see that if $n - \ell(\mu) = k$, then

$$a_\mu(p_k) = \delta_{\mu,(k+1,1^{n-k-1})}.$$

We could not obtain any explicit expression for non-leading coefficients $a_\mu(p_k)$ $(n - \ell(\mu) \leq k - 2)$ except $\mu = (1^n)$.

Power-sum sym fun p_k

Theorem [Lascoux-Thibon (2001)]

Coefficients $a_\mu(p_k)$ are of the form

$$a_\mu(p_k) = \sum_{i=0}^{m_1(\mu)} c^k_{\overline{\mu}\cup(1^i)} \binom{m_1(\mu)}{i},$$

where $\overline{\mu}$ is obtained from μ by erasing its parts equal to 1 (i.e. $\mu = \overline{\mu} \cup (1^{m_1(\mu)})$). The c^k_ρ have the genarating function

$$\sum_{k=0}^{\infty} c^k_\rho \frac{t^k}{k!} = \frac{e^{-t}}{|\rho|!}(1-e^{-t})^{|\rho|-2} \prod_{r\geq 1}(e^{rt}-1)^{m_r(\rho)}.$$

Power-sum sym fun p_k

Corollary [Fujii-Kanno-Moriyama-Okada (2008)]

The coefficient of the identity permutation in $p_{2k}(J_1, \dots, J_n)$ is given by

$$a_{(1^n)}(p_{2k}) = \sum_{r=1}^{k} \binom{n}{r+1} \frac{2^r}{(r+1)!} \sum_{\substack{k_1,k_2,\dots,k_r \geq 1 \\ k_1+k_2+\cdots+k_r=k}} \frac{(2k)!}{(2k_1)!(2k_2)!\cdots(2k_r)!}.$$

Note: $a_{(1^n)}(p_{2k+1}) = 0$.

Example

$$p_2(J_1, J_2, \dots, J_n) = 1\, C_{(3,1^{n-3})} + \binom{n}{2} C_{(1^n)}.$$

$$p_3(J_1, J_2, \dots, J_n) = 1\, C_{(4,1^{n-4})} + \left[2\binom{n-2}{1} + 1\right] C_{(2,1^{n-2})}.$$

$$p_4(J_1, J_2, \dots, J_n) = 1\, C_{(5,1^{n-5})} + \left[3\binom{n-3}{1} + 5\right] C_{(3,1^{n-3})} + 4\, C_{(2^2,1^{n-4})} + \left[4\binom{n}{3} + \binom{n}{2}\right] C_{(1^n)}.$$

Monomial sym fun m_λ

Consider

$$m_\lambda(J_1, J_2, \dots, J_n) = \sum_{(\alpha_1,\alpha_2,\dots,\alpha_n)} J_1^{\alpha_1} J_2^{\alpha_2} \cdots J_n^{\alpha_n}$$

summed over all distinct permutations $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of $(\lambda_1, \lambda_2, \dots, \lambda_n)$.

ex.

$$m_{(2,1)}(J_1, J_2, \dots, J_n) = \sum_{k \neq l} J_k^2 J_l.$$

$$m_{(2,2)}(J_1, J_2, \dots, J_n) = \sum_{k < l} J_k^2 J_l^2.$$

Monomial sym fun m_λ

Theorem [M-Novak (2009 preprint)]

The leading coefficients are given as follows: If $n - \ell(\mu) = |\lambda|$, then

$$a_\mu(m_\lambda) = \sum_{(\lambda^{(1)},\lambda^{(2)},\dots)} \mathrm{RC}(\lambda^{(1)})\mathrm{RC}(\lambda^{(2)})\cdots$$

summed over all sequences of partitions such that

$$\lambda^{(i)} \vdash \mu_i - 1 \ (i \geq 1) \quad \text{and} \quad \lambda = \lambda^{(1)} \cup \lambda^{(2)} \cup \cdots .$$

Here the numbar $\mathrm{RC}(\rho)$ is a positive integer defined by

$$\mathrm{RC}(\rho) = \frac{|\rho|!}{(|\rho| - \ell(\rho) + 1)! \prod_{i \geq 1} m_i(\rho)!}.$$

Corollary.

Let $|\lambda| = k$ and $n - \ell(\mu) = k$. Then $a_\mu(m_\lambda)$ is zero unless λ is bigger than $\tilde{\mu} = (\mu_1 - 1, \mu_2 - 1, \dots)$ in dominance partial ordering, i.e.,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \tilde{\mu}_1 + \tilde{\mu}_2 + \cdots + \tilde{\mu}_i \quad \text{for all } i \geq 1.$$

If λ is a hook not equal to (1^k), then

$$a_\mu(m_{(r,1^{k-r})}) = \sum_{i:\mu_i > r} \binom{\mu_i - 1}{r}.$$

Example

$$m_{(2,1)}(J_1, J_2, \dots, J_n) = 3C_{(4,1^{n-4})} + 1C_{(3,2,1^{n-5})} + \left[\binom{n-2}{2} + 2\binom{n-2}{1} \right] C_{(2,1^{n-2})}.$$

$$m_{(3,1)}(J_1, J_2, \dots, J_n) = 4C_{(5,1^{n-5})} + 1C_{(4,2,1^{n-6})}$$
$$+ \left[6\binom{n-3}{1} + 4 \right] C_{(3,1^{n-3})} + \left[4\binom{n-4}{1} + 10 \right] C_{(2^2,1^{n-4})} + 2\binom{n}{3} C_{(1^n)}.$$

$$m_{(2^2)}(J_1, J_2, \dots, J_n) = 2C_{(5,1^{n-5})} + 1C_{(2^2,1^{n-4})}$$
$$+ \left[\binom{n-3}{2} + 3\binom{n-3}{1} + 1 \right] C_{(3,1^{n-3})} + 2C_{(2^2,1^{n-4})} + \left[3\binom{n}{4} + 2\binom{n}{3} \right] C_{(1^n)}.$$

Hyperoctahedral group

Let H_n be the subgroup of S_{2n} generated by

$$\begin{aligned} (2k-1\ 2k) & \qquad (1 \le k \le n) \\ (2i-1\ 2j-1)(2i\ 2j) & \qquad (1 \le i < j \le n). \end{aligned}$$

Let $\mathcal{H}_n$ be a subset of $\mathbb{C}[S_{2n}]$ given by

$$\mathcal{H}_n := \{f : S_{2n} \to \mathbb{C} \mid f(\tau_1 g \tau_2) = f(g)\ (g \in S_{2n},\ \tau_1, \tau_2 \in H_n)\}.$$

This is a commutative algebra, called the Hecke algebra for the Gelfand pair, or the double coset algebra.

Orthogonal Weingarten function

Fact. [Collins-M (2009)], [M (2010)]

Let $N \geq 2n-1$. The orthogonal Weingarten function defined by

$$\mathrm{Wg}_n^{O(N)} = \sum_{k=0}^{\infty} (-1)^k N^{-n-k} h_k(J_1, J_3, \ldots, J_{2n-1}) \cdot P_n$$

is applied to orthogonal matrix integrals. Here

$$P_n = \sum_{\tau \in H_n} \tau.$$

We study $F(J_1, J_3, \ldots, J_{2n-1})P_n$ (where F is a symmetric function).

Recall that $Z(\mathbb{C}[S_n])$ is generated by $F(J_1, J_2, \dots, J_n)$ where the F are symmetric functions.

Theorem [M (2010)]

For any symmetric function F,

$$F(J_1, J_3, \dots, J_{2n-1})P_n$$

belongs to $\mathcal{H}_n$.

The following theorem was conjectured in [M (2010)].

Theorem [Aker-Can (2010 preprint)]

$$\mathcal{H}_n = \{F(J_1, J_3, \dots, J_{2n-1})P_n \mid F \in \Lambda\}.$$

Zonal spherical functions

Recall that the class algebra $Z(\mathbb{C}[S_n])$ has two bases $\{\chi^\lambda\}$ and $\{C_\mu\}$. Similarly, the double coset algebra $\mathcal{H}_n$ has two bases $\{\omega^\lambda\}$ and $\{C'_\mu\}$.

$$\omega^\lambda(\sigma) = \frac{1}{2^n n!} \sum_{\tau \in H_n} \chi^{2\lambda}(\sigma\tau) \quad (\sigma \in S_{2n})$$

$C'_\mu = \sum \sigma$ summed over all permutations in S_{2n} of coset-type μ.

Note: Double cosets $H_n \sigma H_n$ $(\sigma \in S_{2n})$ are parametrized by partitions μ of n.

We call ω^λ zonal spherical functions and C'_μ double coset sums.

Spherical expansions

Recall

$$F(J_1, J_2, \ldots, J_n) = \sum_{\lambda \vdash n} F(A_\lambda) \frac{f^\lambda}{n!} \chi^\lambda$$

where $A_\lambda = \{j - i \mid (i,j) \in \lambda\}$.

Theorem [M (2010)]

$$F(J_1, J_3, \ldots, J_{2n-1}) P_n = \sum_{\lambda \vdash n} F(A'_\lambda) \frac{f^{2\lambda}}{(2n-1)!!} \omega^\lambda$$

where $A'_\lambda = \{2j - i - 1 \mid (i,j) \in \lambda\}$.

Problem

Recall that we defined $a_\mu(F)$ by

$$F(J_1, J_2, \ldots, J_n) = \sum_{\mu \vdash n} a_\mu(F) C_\mu.$$

Now we define $b_\mu(F)$ by

$$F(J_1, J_3, \ldots, J_{2n-1}) P_n = \sum_{\mu \vdash n} b_\mu(F) C'_\mu.$$

Problem

Evaluate $b_\mu(F)$.

General properties

Let F be a homogeneous symmetric function of degree k.

Theorem [M (2010)]

1. $b_\mu(F)$ is zero unless $n - \ell(\mu) \leq k$.
2. $b_\mu(m_\lambda) \geq a_\mu(m_\lambda)$ for all λ, μ.
3. If $n - \ell(\mu) = k$ then $b_\mu(F) = a_\mu(F)$. In particular, they are independent of n. (We call $\{b_\mu(F) \mid n - \ell(\mu) = k\}$ leading coefficients.)

We call $\{b_\mu(F) \mid n - \ell(\mu) = k - 1\}$ subleading coefficients. Recall that all $a_\mu(F)$ are zero if $n - \ell(\mu) = k - 1$.

Example

$$h_3(J_1, J_2, \ldots, J_n) = 5\, C_{(4,1^{n-4})} + 2\, C_{(3,2,1^{n-5})} + 1\, C_{(2^3,1^{n-6})} + \left[\binom{n-2}{2} + 4\binom{n-2}{1} + \binom{n-2}{0} \right] C_{(2,1^{n-2})}.$$

$$h_3(J_1, J_3, \ldots, J_{2n-1}) P_n = 5\, C'_{(4,1^{n-4})} + 2\, C'_{(3,2,1^{n-5})} + 1\, C'_{(2^3,1^{n-6})} + 6\, C'_{(3,1^{n-3})} + 2\, C'_{(2^2,1^{n-4})} + \left[2\binom{n-2}{2} + 8\binom{n-2}{1} + 3\binom{n-2}{0} \right] C'_{(2,1^{n-2})} + 2\binom{n}{2} C'_{(1^n)}.$$

Subleading coefficients for h_k

The following theorem was conjectured in [M (2010)].

Theorem [Féray (2010 preprint)]

Suppose $n - \ell(\mu) = k - 1$ and suppose μ is a hook. (Hence μ must be $(k+1, 1^{n-k-1})$.) Then

$$b_{(k+1,1^{n-k-1})}(h_k) = 4^k - \binom{2k+1}{k}.$$

More generally, each subleading coefficient $b_\mu(h_k)$ coincides with the number of Dyck paths of type μ.

Complete sym fun h_k

Theorem [Féray (2010 preprint)]

(Recall that) coefficients $a_\mu^k := a_\mu(h_k)$ satisfy the following reccurance identity:

$$a^k_{\rho\cup(m)} = \delta_{m,1} a^k_\rho + \sum_{i=1}^{\ell(\rho)} \rho_i a^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{r=1}^{m-1} a^{k-1}_{\rho\cup(r)\cup(m-r)}.$$

Coefficients $b_\mu^k := b_\mu(h_k)$ satisfy

$$b^k_{\rho\cup(m)} = \delta_{m,1} b^k_\rho + 2\sum_{i=1}^{\ell(\rho)} \rho_i b^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{r=1}^{m-1} b^{k-1}_{\rho\cup(r)\cup(m-r)} + (m-1) b^{k-1}_{\rho\cup(m)}.$$

Open problems

Let F be a homogeneous symmetric function of degree k.

Conjecture. [M (2010), Féray (2010 preprint)]

1. Subleading coefficients $b_\mu(F)$ ($\mu \vdash n$ and $n - \ell(\mu) = k - 1$) are independent of n.
2. As $a_\mu(m_\lambda)$ are so, for all $\mu \vdash n$, coefficients $b_\mu(m_\lambda)$ are of the form

$$b_\mu(m_\lambda) = \sum_{i=0}^{m_1(\mu)} d_i \binom{m_1(\mu)}{i}$$

with non-negative integers d_i.

Jack extension

Let α be a positive real number. We can define $a_\mu^{(\alpha)}(F)$ by

$$F(A_\lambda^{(\alpha)}) = \sum_{\mu \vdash n} a_\mu^{(\alpha)}(F)\theta_\mu^{(\alpha)}(\lambda), \qquad J_\lambda^{(\alpha)} = \sum_{\mu \vdash n} \theta_\mu^{(\alpha)}(\lambda) p_\mu.$$

Here $J_\lambda^{(\alpha)}$ are Jack functions and

$$A_\lambda^{(\alpha)} = \{(j-1) - (i-1)/\alpha \mid (i,j) \in \lambda\}.$$

Then

$$a_\mu^{(1)}(F) = a_\mu(F), \qquad a_\mu^{(2)}(F) = b_\mu(F).$$

Jack extension

Proposition.

For any $\alpha > 0$, $a_\mu^{(\alpha)}(e_k) = 1$ if $|\mu| - \ell(\mu) = k$, and $a_\mu^{(\alpha)}(e_k) = 0$ otherwise.

Conjecture [M (2010)]

Let F be a homogeneous symmetric function of degree k.

1. (leading coefficients.) If $n - \ell(\mu) = k$, then $a_\mu^{(\alpha)}(F) = a_\mu(F)$. Therefore those coefficients are independent of α and n.
2. (subleading coefficients.) If $n - \ell(\mu) = k - 1$, then $a_\mu^{(\alpha)}(F) = (\alpha - 1) b_\mu(F)$. Therefore those coefficients are independent of n.