Computations of unitary matrix integrals — some works related to Riemann zeta and Juchys-Murphy elements

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Unitary matrix

The unitary group:

$$U(N) = \{U = (U_{ij})_{1 \le i,j \le N} \in GL(N, \mathbb{C}) \mid UU^* = I\}.$$

Normalized Haar measure $\mathrm{d}U$:

$$\mathrm{d}(V_1 U V_2) = \mathrm{d}U \qquad \text{for fixed } V_1, V_2 \in U(N).$$

$$\int_{U(N)} 1 \mathrm{d}U = 1.$$

Weyl's integration formula

Theorem (Weyl)

For any class function f on $U(N)$,

$$\int_{U(N)} f(U)\mathrm{d}U = \int_{[0,2\pi)^N} f(\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})) \times p_N(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})\mathrm{d}\theta_1\mathrm{d}\theta_2 \cdots \mathrm{d}\theta_N$$

where

$$p_N(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}) = \frac{1}{(2\pi)^N N!} \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

Hence the joint density function for eigenvalues of a random unitary matrix U is given by p_N.

"Zeta" on the unitary group

Consider the Characteristic polynomial

$$Z_U(s) := \det(I - Us) \qquad (U \in U(N),\ s \in \mathbb{C}).$$

Dirichlet series representation: $Z_U(s) = \sum_{n=0}^{N} a_n s^n$.
Product expression: $Z_U(s) = \prod_{j=1}^{N} (1 - e^{i\theta_j} s)$.
Functional equation: $Z_U(s) = (\det U)(-s)^N Z_{U^*}(s^{-1})$.
Zeros: eigenvalues of U.
Location of zeros: the unit circle.
Determinantal expression: the definition itself.

Montgomery conjecture

Pair correlation: Riemann zeta

Given real numbers $a < b$, put

$$D_{(a,b)}(T) = \# \left\{ (\gamma, \gamma') \;\middle|\; \begin{array}{c} \zeta(\frac{1}{2} + i\gamma) = \zeta(\frac{1}{2} + i\gamma') = 0, \\ 0 < \gamma, \gamma' \le T, \\ \frac{2\pi a}{\log T} \le \gamma - \gamma' \le \frac{2\pi b}{\log T} \end{array} \right\}$$

Conjecture [Montgomery (1973)]

For fixed $0 < a < b < \infty$,

$$\lim_{T \to \infty} \frac{2\pi}{T \log T} D_{(a,b)}(T) \stackrel{?}{=} \int_a^b \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) \mathrm{d}u.$$

Pair correlation: unitary matrix

Consider eigenvalues of a random unitary matrix from $U(N)$.
The pair correlation function

$$R_{2,N}(\phi,\psi) := \frac{N!}{(N-2)!}\int_{[0,2\pi)^{N-2}} p_N(e^{i\phi}, e^{i\psi}, e^{i\theta_3}, \dots, e^{i\theta_N})\mathrm{d}\theta_3 \cdots \mathrm{d}\theta_N.$$

This is the probability that two eigenvalues exist around the points $e^{i\phi}, e^{i\psi}$.
Then

$$R_{2,N}(\phi,\psi) = \left(\frac{N}{2\pi}\right)^2 - \left(\frac{\sin(N(\phi-\psi)/2)}{2\pi\sin((\phi-\psi)/2)}\right)^2.$$

Hence, fixed $\xi, \eta > 0$,

$$\lim_{N\to\infty}\left(\frac{2\pi}{N}\right)^2 R_{2,N}\left(\frac{2\pi\xi}{N}, \frac{2\pi\eta}{N}\right) = 1 - \left(\frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)}\right)^2.$$

Keating-Snaith conjecture

Keating-Snaith conjecture

Conjecture [Keating-Snaith (2000)]

For each positive integer m,

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{m^2}} \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2m} \mathrm{d}t \overset{?}{=} a(m) f_U(m).$$

where

$$a(m) = \prod_{p:\mathrm{prime}} \left[(1-p^{-1})^{m^2} \sum_{k=0}^{\infty} \binom{m+k-1}{k}^2 p^{-k} \right],$$

$$f_U(m) = \prod_{j=0}^{m-1} \frac{j!}{(m+j)!}.$$

Keating-Snaith conjecture

The Keating-Snaith conjecture is proved for only $m = 1$ and $m = 2$.
[Hardy-Littlewood (1918)]: $a(1) f_U(1) = 1 \times 1 = 1$.

$$\frac{1}{T} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 \mathrm{d}t \sim \log T.$$

[Ingham (1926)]: $a(2) f_U(2) = \frac{1}{\zeta(2)} \times \frac{1}{2!3!} = \frac{1}{2\pi^2}$.

$$\frac{1}{T} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^4 \mathrm{d}t \sim \frac{1}{2\pi^2} (\log T)^4.$$

Moments of the characteristic polynomial

Theorem [Keating-Snaith (2000)]

For fixed $|s| = 1$,

$$\int_{U(N)} |\det(I - Us)|^{2m} \mathrm{d}U = \prod_{j=1}^{N-1} \frac{j!(j+2m)!}{((j+m)!)^2}.$$

Hence

$$\lim_{N\to\infty} \frac{1}{N^{m^2}} \int_{U(N)} |\det(I - Us)|^{2m} \mathrm{d}U = f_U(m) = \prod_{j=0}^{m-1} \frac{j!}{(m+j)!}.$$

A problem

Consider an integrable function f on $U(N)$. We would like to compute an integral

$$\int_{U(N)} f(U)\mathrm{d}U.$$

We can apply Weyl's integration formula if f is central.

How can we compute an integral for non-central f ?

ex. a moment of a principal minor of the characteristic polynomial

$$f(U) = \left|\det(\delta_{ij} - su_{ij})_{1\le i,j\le k}\right|^{2m}.$$

In general, we would like to compute an integral for a polynomial in u_{ij} and $\bar{u}_{ij}$.

Weingarten calculus

$U = (u_{ij})$: a Haar-distributed random unitary matrix from $U(N)$.

Proposition.

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_m j'_m}} \mathrm{d}U = 0$$

unless $n = m$.

Proposition.

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_n j'_n}} \mathrm{d}U = 0$$

unless $(i_1, \dots, i_n)$ and $(j_1, \dots, j_n)$ are rearrangements of $(i'_1, \dots, i'_n)$ and $(j'_1, \dots, j'_n)$, respectively.

Example

$$\int_{U(N)} u_{11} u_{12}^2 \overline{u_{12} u_{22}} \mathrm{d}U = 0.$$

$$\int_{U(N)} u_{11} u_{12} u_{12} \overline{u_{11} u_{12} u_{22}} \mathrm{d}U = 0.$$

row index : $(1, 1, 1)$ and $(1, 1, 2)$
column index : $(1, 2, 2)$ and $(1, 2, 2)$.

Weingarten calculus

Theorem [Collins (2003)]

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_n j'_n}} \mathrm{d}U$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma \cdot (i_1, \dots, i_n) = (i'_1, \dots, i'_n)}} \sum_{\substack{\tau \in S_n \\ \tau \cdot (j_1, \dots, j_n) = (j'_1, \dots, j'_n)}} \mathrm{Wg}_n^{U(N)}(\sigma^{-1}\tau).$$

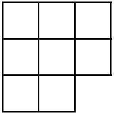
Here the function $\mathrm{Wg}_n^{U(N)}(\sigma)$ is given by

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \in \mathbb{Y}_n} \frac{f^\lambda}{\prod_{(i,j)\in\lambda}(N+j-i)} \chi^\lambda(\sigma) \qquad (\sigma \in S_n),$$

which is called the unitary Weingarten function.

Representation theory

$\mathbb{Y}_n$: the set of Young diagram with n boxes.



$\mathbb{Y}_n \cong \{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l) \mid \lambda_i \in \mathbb{Z}_{>0}, |\lambda| := \sum_i \lambda_i = n\}$.
Irreducible representations of S_n are parametrized by elements in $\mathbb{Y}_n$.
χ^λ: the irreducible character corresponding to λ.
f^λ: the dimension of the irrep corresponding to λ.

Example. $n = 2$.

$$\mathrm{Wg}_2^{U(N)}(\sigma) = \begin{cases} \frac{1}{(N+1)(N-1)} & \sigma = \mathrm{id}_2 \\ \frac{-1}{N(N+1)(N-1)} & \sigma = (1\ 2) \end{cases}$$

Example of Weingarten calculus

$$\int_{U(N)} u_{11} u_{12} u_{12} \overline{u_{11} u_{12} u_{12}} \mathrm{d}U$$

$$= \sum_{\sigma \in S_3} \sum_{\tau \in \{\mathrm{id}_3, (2\ 3)\}} \mathrm{Wg}_3^{U(N)}(\sigma^{-1}\tau)$$

$$= 2 \sum_{\sigma \in S_3} \mathrm{Wg}_3^{U(N)}(\sigma) = \frac{2}{N(N+1)(N+2)}.$$

$$\int_{U(N)} u_{11} u_{22} u_{33} \overline{u_{12} u_{23} u_{31}} \mathrm{d}U$$

$$= \mathrm{Wg}_3^{U(N)}((1\ 2\ 3)) = \frac{2}{(N+2)(N+1)N(N-1)(N-2)}.$$

Example of Weingarten calculus

Proposition [Novak (2007)]

$$\int_{U(N)} |u_{11} + u_{22} + \cdots + u_{kk}|^{2n} \mathrm{d}U = \sum_{\lambda \in \mathbb{Y}_n : \ell(\lambda) \le k} (f^\lambda)^2 \prod_{(i,j) \in \lambda} \frac{k+j-i}{N+j-i}.$$

In particular,

$$\int_{U(N)} |u_{ij}|^{2n} \mathrm{d}U = \frac{n!}{N(N+1)\cdots(N+n-1)}.$$

Hence the random variable $\sqrt{N} u_{ij}$ converges in distribution to a standard complex Gaussian random variable.

Example of Weingarten calculus

Let $Z_k(x; U) = \det(x\delta_{ij} + u_{ij})_{1 \le i,j \le k}$.

Proposition.

For any complex numbers $x, y \in \mathbb{C}$,

$$\int_{U(N)} Z_k(x; U) Z_k(y; U^*) \mathrm{d}U = \binom{N}{k}^{-1} \sum_{r=0}^{k} \binom{N-k+r}{r} (xy)^r.$$

Group algebra

$$\mathbb{C}[S_n] := \left\{ \sum_{\sigma \in S_n} f(\sigma)\sigma \;\middle|\; f(\sigma) \in \mathbb{C} \right\}$$

$$L^1(S_n, \mathbb{C}) := \{f : S_n \to \mathbb{C} \mid \text{a function}\}$$

Product

$$(\sum_{\sigma} f(\sigma)\sigma)(\sum_{\tau} g(\tau)\tau) = \sum_{\sigma,\tau} f(\sigma)g(\tau)\sigma\tau$$

$$= \sum_{\sigma} \left(\sum_{\tau} f(\sigma\tau^{-1})g(\tau) \right) \sigma.$$

Jucys-Murphy elements

Consider the following elements in $\mathbb{C}[S_n]$.

$$
\begin{aligned}
J_1 &= 0, \\
J_2 &= (1\ 2). \\
J_3 &= (1\ 3) + (2\ 3). \\
J_4 &= (1\ 4) + (2\ 4) + (3\ 4). \\
&\dots \\
J_n &= (1\ n) + (2\ n) + (3\ n) + \cdots + (n-1\ n).
\end{aligned}
$$

They are commute.

Weingarten and Jucys-Murphy

Theorem [M-Novak (2009 preprint)]

Suppose $N \geq n$. In $\mathbb{C}[S_n]$,

$$\mathrm{Wg}_n^{U(N)} = \sum_{k=0}^{\infty} (-1)^k N^{-n-k} h_k(J_1, J_2, \dots, J_n),$$

where $h_k(x_1, x_2, \dots, x_n)$ is the k-th complete homogeneous symmetric polynomial

$$h_k(x_1, x_2, \dots, x_n) = \sum_{\substack{a_1, a_2, \dots, a_n \geq 0 \\ a_1 + a_2 + \cdots + a_n = k}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Expressions for Weingarten functions

(1)

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \int_{U(N)} \left(\prod_{i=1}^{n} u_{ii} \overline{u_{i\sigma(i)}} \right) \mathrm{d}U.$$

(2) In $\mathbb{C}[S_n]$, the element $\mathrm{Wg}_n^{U(N)}$ is the inverse of the function $S_n \ni \sigma \mapsto N^{(\text{Number of cycles in } \sigma)}$.

(3)

$$\mathrm{Wg}_n^{U(N)} = \frac{1}{n!} \sum_{\lambda \in \mathbb{Y}_n} \frac{f^\lambda}{\prod_{(i,j)\in\lambda}(N+j-i)} \chi^\lambda.$$

(4)

$$\mathrm{Wg}_n^{U(N)} = \sum_{k=0}^{\infty} (-1)^k N^{-n-k} h_k(J_1, J_2, \dots, J_n).$$

Asymptotic behavior

Theorem [M-Novak (2009 preprint)]

Suppose $N \geq n$. Then

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \sum_{k \geq 0} (-1)^k \frac{a_k(\sigma)}{N^{n+k}},$$

where $a_k(\sigma)$ is the multiplicity of σ in $h_k(J_1, \dots, J_n)$.
In other words, $a_k(\sigma)$ coincides with the number of sequences $(s_1, t_1, s_2, t_2, \dots, s_k, t_k) \in \{1, 2, \dots, n\}^{\times 2k}$ satisfying

- $s_i < t_i$ for all i.
- $1 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq n$.
- $\sigma = (s_1\ t_1)(s_2\ t_2) \cdots (s_k\ t_k)$.

Asymptotic behavior

Corollary [M-Novak (2009 preprint)]

Suppose $N \geq n$. If $\sigma \in S_n$ is of cycle-type $\mu \in \mathbb{Y}_n$, then

$$\mathrm{Wg}_n^{U(N)}(\sigma) = (-1)^{n-\ell(\mu)} \sum_{k=0}^{\infty} \frac{a_{n-\ell(\mu)+2k}(\sigma)}{N^{2n-\ell(\mu)+2k}}$$

with

$$a_{n-\ell(\mu)}(\sigma) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}(\mu_i - 1), \qquad \mathrm{Cat}(m) = \frac{(2m)!}{(m+1)!m!}$$

and non-negative integers $a_{n-\ell(\mu)+2k}(\sigma), k = 1, 2, \dots$.

Example

$$\mathrm{Wg}_n^{U(N)}(\mathrm{id}_n) = \int_{U(N)} |u_{11} u_{22} \cdots u_{nn}|^2 \mathrm{d}U$$

$$= \sum_{k=0}^{\infty} \frac{a_{2k}(\mathrm{id}_n)}{N^{n+2k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{N^{n+2k}} [\mathrm{id}_n] h_{2k}(J_1, \dots, J_n)$$

$$= N^{-n} + \binom{n}{2} N^{-n-2} + \left[3\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2} \right] N^{-n-4} + O(N^{-n-6}).$$

Orthogonal group

$$
\begin{aligned}
O(N) =& \{V = (V_{ij})_{1 \le i,j \le N} \in GL(N, \mathbb{R}) \mid VV^t = I\}. \\
SO(N) =& \{V \in O(N) \mid \det(V) = 1\}.
\end{aligned}
$$

Theorem [Keating-Snaith (2000)]

$$
\begin{aligned}
\int_{SO(2N)} \det(I+V)^m \mathrm{d}V =& 2^m \prod_{j=0}^{m-1} \frac{(2N+2j-1)!}{2^{j-1}(2j-1)!!(2N+j-1)!} \\
\sim& \frac{2^m}{\prod_{j=1}^{m-1}(2j-1)!!} N^{m(m-1)/2}.
\end{aligned}
$$

Put $f_O(m) = 2^m \left(\prod_{j=1}^{m-1}(2j-1)!!\right)^{-1}$.

Moments of L-functions

Let $H_2(N)$ be the collection of Hecke newforms of weight 2 for the congruence group $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$. Denote by $L_f(s)$ the L-function associated with $f \in H_2(N)$.

Conjecture [Keating-Snaith (2000)]

$$\lim_{N\to\infty} \frac{1}{(\log N)^{m(m-1)/2}|H_2(N)|} \sum_{f\in H_2(N)} L_f\left(\frac{1}{2}\right)^m \stackrel{?}{=} a(m)f_O(m)/2,$$

where

$$a(m) = \prod_{p\nmid N}(1-p^{-1})^{m(m-1)/2} \times \frac{2}{\pi}\int_0^\pi \sin^2\theta \left(\frac{e^{i\theta}(1-e^{i\theta}/\sqrt{p})^{-1} - e^{-i\theta}(1-e^{-i\theta}/\sqrt{p})^{-1}}{e^{i\theta}-e^{-i\theta}}\right)^m \mathrm{d}\theta.$$

Weingarten calculus

Let $V = (v_{ij})_{1 \le i,j \le N}$ be a random orthogonal matrix.

$$\int_{O(N)} v_{i_1 j_1} v_{i_2 j_2} \cdots v_{i_n j_n} \mathrm{d}V = 0$$

unless n is even.

$$\int_{O(N)} v_{i_1 j_1} v_{i_2 j_2} \cdots v_{i_{2n} j_{2n}} \mathrm{d}V = 0$$

unless there exist pairings $\mathfrak{m}, \mathfrak{n}$ on $\{1, 2, \dots, 2n\}$ such that

$$\{p, q\} \in \mathfrak{m} \quad \Rightarrow \quad i_p = i_q.$$
$$\{p, q\} \in \mathfrak{n} \quad \Rightarrow \quad j_p = j_q.$$

Weingarten calculus

Let $\mathcal{M}(2n)$ be the set of all pairings on $\{1, 2, \ldots, 2n\}$.

[Collins-Śniady (2006)]

$$\int_{O(N)} v_{i_1 j_1} v_{i_2 j_2} \cdots v_{i_{2n} j_{2n}} \mathrm{d}V = \sum_{\mathfrak{m}} \sum_{\mathfrak{n}} \mathrm{Wg}_n^{O(N)}(\mathfrak{m}, \mathfrak{n})$$

summed over all pairings $\mathfrak{m}, \mathfrak{n}$ in $\mathcal{M}(2n)$ such that

$$\{p, q\} \in \mathfrak{m} \quad \Rightarrow \quad i_p = i_q.$$
$$\{p, q\} \in \mathfrak{n} \quad \Rightarrow \quad j_p = j_q.$$

We will give the definition of the orthogonal Weingarten function $\mathrm{Wg}_n^{O(N)}$ later.

Example

- $$\int_{O(N)} v_{11} v_{12} v_{23} v_{21} \, \mathrm{d}V = 0.$$
- $$\int_{O(N)} v_{11} v_{12} v_{32} v_{31} \, \mathrm{d}V$$
$$= \mathrm{Wg}_2^{O(N)}(\; \{\{1,2\},\{3,4\}\} \; , \; \{\{1,4\},\{2,3\}\} \;)$$
$$= \frac{-1}{N(N+2)(N-1)}.$$
- $$\int_{O(N)} v_{11}^{2n} \mathrm{d}V$$
$$= \sum_{\mathfrak{m},\mathfrak{n} \in \mathcal{M}(2n)} \mathrm{Wg}_n^{O(N)}(\mathfrak{m},\mathfrak{n}) = \frac{(2N-1)!!}{N(N+2)(N+4)\cdots(N+2n-2)}.$$

Hyperoctahedral group

Let H_n be the subgroup of S_{2n} generated by

$$\begin{aligned} &(2k-1\ 2k) && (1 \le k \le n) \\ &(2i-1\ 2j-1)(2i\ 2j) && (1 \le i < j \le n). \end{aligned}$$

Each pairing $\mathfrak{m}$ can be regarded as permutations in S_{2n} by

$$\begin{aligned} \mathfrak{m} &= \{\{\mathfrak{m}(1), \mathfrak{m}(2)\}, \dots, \{\mathfrak{m}(2n-1), \mathfrak{m}(2n)\}\} \\ &\quad (\mathfrak{m}(2i-1) < \mathfrak{m}(2i), \quad \mathfrak{m}(1) < \mathfrak{m}(3) < \cdots < \mathfrak{m}(2n-1)) \\ &\mapsto \begin{pmatrix} 1 & 2 & \cdots & 2n \\ \mathfrak{m}(1) & \mathfrak{m}(2) & \cdots & \mathfrak{m}(2n) \end{pmatrix} \end{aligned}$$

Note $S_{2n} = \bigsqcup_{\mathfrak{m} \in \mathcal{M}(2n)} \mathfrak{m} H_n$,

Orthogonal Weingarten function

(1) [Collins-Śniady (2006)] Putting $\sigma = \mathfrak{m}^{-1}\mathfrak{n} \in S_{2n}$,

$$\mathrm{Wg}^{O(N)}(\mathfrak{m}, \mathfrak{n}) = \int_{O(N)} \prod_{k=1}^{n} v_{k,[\sigma(2k-1)/2]} v_{k,[\sigma(2k)/2]} \mathrm{d}V.$$

(2) [Collins-Śniady (2006)]
The matrix $\mathrm{Wg}_n^{O(N)} = (\mathrm{Wg}_n^{O(N)}(\mathfrak{m}, \mathfrak{n}))_{\mathfrak{m},\mathfrak{n} \in \mathcal{M}(2n)}$ is the inverse of the matrix

$$G_n^{O(N)} = (N^{|\mathfrak{m} \vee \mathfrak{n}|})_{\mathfrak{m},\mathfrak{n} \in \mathcal{M}(2n)}.$$

Here $|\mathfrak{m} \vee \mathfrak{n}|$ is the number of blocks of a set-partition $\mathfrak{m} \vee \mathfrak{n}$, (By definition, each block $\{p, q\}$ in $\mathfrak{m}$ and $\mathfrak{n}$ is included in a block of $\mathfrak{m} \vee \mathfrak{n}$,

Orthogonal Weingarten function

Put $\sigma = \mathfrak{m}^{-1}\mathfrak{n} \in S_{2n}$ and let $\mathrm{Wg}_n^{O(N)}(\sigma) = \mathrm{Wg}_n^{O(N)}(\mathfrak{m}, \mathfrak{n})$.
The 3rd expression.

Theorem [Collins-M (2009)]

$$\mathrm{Wg}_n^{O(N)}(\sigma) = \frac{1}{(2n-1)!!} \sum_{\lambda \in \mathbb{Y}_n} \frac{f^{2\lambda}}{\prod_{(i,j)\in\lambda}(N+2j-i-1)} \omega^{\lambda}(\sigma)$$

where ω^{λ} is the zonal spherical function for the Gelfand pair (S_{2n}, H_n), given by

$$\omega^{\lambda}(\sigma) = \frac{1}{|H_n|} \sum_{\tau \in H_n} \chi^{2\lambda}(\sigma\tau) \qquad (\sigma \in S_{2n}).$$

Orthogonal Weingarten function

The 4th expression.

Theorem [M (2010)]

$$\mathrm{Wg}_n^{O(N)} = \sum_{k=0}^{\infty} (-1)^k N^{-n-k} h_k(J_1, J_3, \dots, J_{2n-1}) \cdot P_n,$$

where $h_k(x_1, x_2, \dots, x_n)$ is the k-th complete homogeneous symmetric polynomial and

$$P_n = \sum_{\tau \in H_n} \tau.$$

Example

$$\mathrm{Wg}_n^{O(N)}(\mathrm{id}_{2n}) = \int_{O(N)} v_{11}^2 v_{22}^2 \cdots v_{nn}^2 \mathrm{d}V$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{N^{n+k}} [\mathrm{id}_{2n}] h_k(J_1, J_3, \dots, J_{2n-1}) P_n$$

$$= N^{-n} - 0N^{-n-1} + n(n-1)N^{-n-2} - n(n-1)N^{-n-3} + O(N^{-n-4}).$$