Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Jacobi-Trudi formula for Jack functions of rectangular shapes and hyperdeterminants

Sho Matsumoto

ABSTRACT. We derive a Jacobi-Trudi type formula for Jack functions of rectangular shapes. In this formula, we make use of a hyperdeterminant, which is Cayley's simple generalization of the determinant. In some cases, we also express these Jack functions by a hyperpfaffian, which is a generalization of pfaffians in Cayley's sense. In addition, we give summation identities of the products of Schur functions involving hyperdeterminants and hyperpfaffians.

1. Introduction

The Schur function s_λ [**13**, §I] satisfies the Jacobi-Trudi formula

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n}. \tag{1.1}$$

Here $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition whose length $\ell(\lambda)$ is equal to or less than n, and h_k is the complete symmetric function, or equivalently, the one-row Schur function $h_k = s_{(k)}$.

The Jack function $Q_\lambda^{(\alpha)}$ [**13**, §VI-10] has one parameter $\alpha > 0$ and includes the Schur function as the special case $\alpha = 1$. Furthermore, the Jack function associated with $\alpha = 2$ or $\frac{1}{2}$ is the zonal polynomial related to a Gelfand pair in representation theory.

In this paper, we consider the problem of finding a Jacobi-Trudi type formula for Jack functions. We can see an expansion formula for Jack functions with respect to one-row Jack functions in [**7, 9**]. However, we would like to obtain a Jacobi-Trudi formula expressed in a *determinant-like* form like (1.1). Kerov [**8**] obtained such a formula when λ is a hook. Especially, he gave the expression

$$Q_\lambda^{(\alpha)} = \det\left(\frac{\alpha(\lambda_i - i + j) + \ell(\lambda) - j}{\alpha\lambda_i + \ell(\lambda) - i} g_{\lambda_i - i + j}^{(\alpha)}\right)_{1 \le i,j \le n}, \qquad \text{for a hook } \lambda,$$

where $g_k^{(\alpha)}$ is one-row Jack function $Q_{(k)}^{(\alpha)}$. In non-hook cases, such a Jacobi-Trudi type formula is not known.

Our main result is to give the Jacobi-Trudi type formula for Jack functions associated with *rectangular-shape* partitions. Here we say λ to be rectangular-shape when the associated Young diagram is rectangular, i.e., $\lambda = (L^n) = (L, L, \dots, L)$ for some positive integers L and n.

In our formula, we employ a *hyperdeterminant*. The hyperdeterminant is a simple generalization of the determinant and defined by

$$\det{}^{[2m]}(A) := \frac{1}{n!} \sum_{\sigma_1, \dots, \sigma_{2m} \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1) \cdots \operatorname{sgn}(\sigma_{2m}) \prod_{i=1}^n A(\sigma_1(i), \dots, \sigma_{2m}(i)),$$

for an array $A = (A(i_1, \dots, i_{2m}))_{1 \le i_1, \dots, i_{2m} \le n}$. It is defined by Cayley [**3**] and studied in [**2, 5, 10, 12, 16**].

2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 15A15.
Key words and phrases. Jack function, Jacobi-Trudi formula, hyperdeterminants, pfaffian, Schur function.
The author thanks Research Fellow of the Japan Society for the Promotion of Science, partially supported by Grant-in-Aid for Scientific Research (C) No. 17006193.

Now we give our main theorem. According to the notation of Macdonald's book [**13**], let $P_\lambda^{(\alpha)}$ and $Q_\lambda^{(\alpha)}$ be the Jack P-function and Q-function, respectively. Let e_k be the elementary symmetric function. We put $g_k^{(\alpha)} = e_k = 0$ for negative integers k.

THEOREM 1.1 (Main Theorem). *Let m, n, and L be positive integers. Then we have*

$$Q_{(L^n)}^{(1/m)} = \frac{n!\,(m!)^n}{(mn)!} \cdot \det{}^{[2m]}(g^{(1/m)}_{L+i_1+\cdots+i_m-i_{m+1}-\cdots-i_{2m}})_{1\le i_1,\dots,i_{2m}\le n}, \tag{1.2}$$

$$P_{(n^L)}^{(m)} = \frac{n!\,(m!)^n}{(mn)!} \cdot \det{}^{[2m]}(e_{L+i_1+\cdots+i_m-i_{m+1}-\cdots-i_{2m}})_{1\le i_1,\dots,i_{2m}\le n}. \tag{1.3}$$

This theorem gives the Jacobi-Trudi formula for Jack functions of rectangular shapes with parameter $\alpha = m$ or $1/m$. If we substitute $m = 1$, we obtain the Jacobi-Trudi formula for Schur functions of rectangular shapes,

$$s_{(L^n)} = \det(h_{L+i-j})_{1\le i,j\le n}, \qquad s_{(n^L)} = \det(e_{L+i-j})_{1\le i,j\le n}.$$

The Jacobi-Trudi formula for Jack functions of other partitions is still open.

Furthermore we give another formula for Jack functions. When α is an even integer $2m$ or its reciprocal, we will express the corresponding Jack function of rectangular shapes as a "hyperpfaffian" (see Theorem 4.3). The hyperpfaffian is the pfaffian analogue of the hyperdeterminant. In particular, we obtain the following pfaffian expressions for Jack functions with parameter $\alpha = 2$ or $1/2$.

THEOREM 1.2. *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a rectangular partition of length $\le n$ and let λ' be its conjugate partition. Then we have*

$$Q_\lambda^{(1/2)} = \frac{1}{(2n-1)!!} \cdot \mathrm{pf}((j-i) g_{\gamma_j+2n+1-i-j})_{1\le i<j\le 2n}, \tag{1.4}$$

$$P_{\lambda'}^{(2)} = \frac{1}{(2n-1)!!} \cdot \mathrm{pf}((j-i) e_{\gamma_j+2n+1-i-j})_{1\le i<j\le 2n}, \tag{1.5}$$

with $g_k = g_k^{(1/2)}$ and

$$\gamma_j = \begin{cases} \lambda_{n-j+1}, & \text{for } 1 \le j \le n, \\ \lambda_{j-n}, & \text{for } n+1 \le j \le 2n. \end{cases} \tag{1.6}$$

In particular, expressions in (1.4) *and* (1.5) *are independent of the choice of n.*

EXAMPLE 1.3. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a rectangular partition of length ≤ 3, i.e, λ is $(L, 0, 0)$, $(L, L, 0)$ or (L, L, L) for a positive integer L. Then

$$Q_\lambda^{(1/2)} = \frac{1}{15} \cdot \mathrm{pf}\begin{pmatrix} 0 & g_{\lambda_2+4} & 2g_{\lambda_1+3} & 3g_{\lambda_1+2} & 4g_{\lambda_2+1} & 5g_{\lambda_3} \\ -g_{\lambda_2+4} & 0 & g_{\lambda_1+2} & 2g_{\lambda_1+1} & 3g_{\lambda_2} & 4g_{\lambda_3-1} \\ -2g_{\lambda_1+3} & -g_{\lambda_1+2} & 0 & g_{\lambda_1} & 2g_{\lambda_2-1} & 3g_{\lambda_3-2} \\ -3g_{\lambda_1+2} & -2g_{\lambda_1+1} & -g_{\lambda_1} & 0 & g_{\lambda_2-2} & 2g_{\lambda_3-3} \\ -4g_{\lambda_2+1} & -3g_{\lambda_2} & -2g_{\lambda_2-1} & -g_{\lambda_2-2} & 0 & g_{\lambda_3-4} \\ -5g_{\lambda_3} & -4g_{\lambda_3-1} & -3g_{\lambda_3-2} & -2g_{\lambda_3-3} & -g_{\lambda_3-4} & 0 \end{pmatrix}.$$

□

In the final section, we give another application of hyperdeterminants. We obtain summation identities for the product of Schur functions involving hyperdeterminants and hyperpfaffians.

2. Preliminaries

We use the notation in Macdonald's book [**13**]. We identify a partition λ with the associated Young diagram. Let $\ell(\lambda)$ be the length of λ. Denote by λ' the conjugate partition of λ, or its Young diagram of λ' is obtained by transposing that of λ.

Let $\alpha > 0$ be a positive real number and let $\Lambda(\alpha)$ be the $\mathbb{Q}(\alpha)$-algebra of symmetric functions in variables $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots)$. Let $P_\lambda^{(\alpha)}$ be the Jack P-functions and $Q_\lambda^{(\alpha)}$ the Jack Q-functions, see [**13**, §VI-10]. They have the relation $Q_\lambda^{(\alpha)} = b_\lambda^{(\alpha)} P_\lambda^{(\alpha)}$ with

$$b_\lambda^{(\alpha)} = \prod_{(i,j)\in\lambda} \frac{\alpha(\lambda_i - j) + \lambda'_j - i + 1}{\alpha(\lambda_i - j + 1) + \lambda'_j - i}.$$

They are α-extensions of Schur functions; $P_\lambda^{(1)} = Q_\lambda^{(1)} = s_\lambda$. Note that the one-column Jack P-function $P_{(1^k)}^{(\alpha)}$ is equal to the elementary symmetric function e_k and therefore is independent of α. We set $g_k^{(\alpha)} = Q_{(k)}^{(\alpha)}$. The generating function of $g_k^{(\alpha)}$ is given by

$$G_{\mathbf{x}}^{(\alpha)}(z) = 1 + \sum_{k=1}^{\infty} g_k^{(\alpha)}(\mathbf{x}) z^k = \prod_{i=1}^{\infty} (1 - \mathbf{x}_i z)^{-1/\alpha}.$$

Jack functions satisfy the Cauchy identity

$$\sum_{\lambda:\text{partitions}} P_\lambda^{(\alpha)}(\mathbf{x}) Q_\lambda^{(\alpha)}(\mathbf{y}) = \prod_{i,j=1}^{\infty} (1 - \mathbf{x}_i \mathbf{y}_j)^{-1/\alpha}, \tag{2.1}$$

where $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$ is another sequence of variables.

Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle and let $\mathrm{d}z$ be the Haar measure on $\mathbb{T}$ such that $\int_{\mathbb{T}} \mathrm{d}z = 1$. Jack functions have the orthogonality property

$$\begin{aligned} &\frac{1}{n!}\int_{\mathbb{T}^n} P_\lambda^{(\alpha)}(z_1,\dots,z_n)\overline{Q_\mu^{(\alpha)}(z_1,\dots,z_n)}|V(z_1,\dots,z_n)|^{2/\alpha}\mathrm{d}z_1\dots\mathrm{d}z_n \\ =&\delta_{\lambda\mu} I_n(\alpha) \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - i + 1}{n + j\alpha - i}, \end{aligned} \tag{2.2}$$

where (i,j) in the product runs over all boxes on the Young diagram λ. Here $V(z_1,\dots,z_n)$ is the Vandermonde determinant

$$V(z_1,\dots,z_n) = \prod_{1\le i<j\le n} (z_i - z_j) = \det(z_i^{n-j})_{1\le i,j\le n},$$

and we put

$$I_n(\alpha) = \frac{1}{n!}\int_{\mathbb{T}^n} |V(z_1,\dots,z_n)|^{2/\alpha}\mathrm{d}z_1\dots\mathrm{d}z_n.$$

The explicit value of $I_n(\alpha)$ is obtained from the so-called Dyson conjecture, see e.g. [**1**, §8],

$$I_n(\alpha) = \frac{\Gamma(n/\alpha + 1)}{n!\,\Gamma(1/\alpha + 1)^n}. \tag{2.3}$$

3. Proof of the main theorem

3.1. Hyperdeterminants. We state the definition of the hyperdeterminant again.

Definition 3.1 ([**3, 2**]). For an array $A = (A(i_1,\dots,i_{2m}))_{1\le i_1,\dots,i_{2m}\le n}$, we define the hyperdeterminant of A by

$$\det{}^{[2m]}(A) = \frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_{2m}\in\mathfrak{S}_n} \mathrm{sgn}(\sigma_1)\cdots\mathrm{sgn}(\sigma_{2m}) \prod_{i=1}^{n} A(\sigma_1(i),\dots,\sigma_{2m}(i)).$$

□

We sometimes write A as $(A(i_1,\dots,i_{2m}))_{[n]}$. It is clear that when $m = 1$ Definition 3.1 agrees the definition of the usual determinant of a square matrix.

Let f be a function on $\mathbb{T}$ whose Fourier expansion is $f(z) = \sum_{k\in\mathbb{Z}} d(k) z^k$. We define the *Toeplitz hyperdeterminant* of f by

$$D_n^{[2m]}(f) := \det{}^{[2m]}(d(i_1 + i_2 + \dots + i_m - i_{m+1} - \dots - i_{2m}))_{[n]}.$$

REMARK 3.2. Luque and Thibon [**12**] studied the Hankel hyperdeterminant

$$H_n^{[2m]}(f) = \det{}^{[2m]}(d(i_1 + i_2 + \cdots + i_m))_{0 \le i_1, \dots, i_{2m} \le n-1}.$$

Any Toeplitz hyperdeterminant can be expressed as a Hankel hyperdeterminant. In fact, it is easy to see that

$$D_n^{[2m]}(f) = (-1)^{mn(n-1)/2} H_n^{[2m]}(z^{-m(n-1)} f(z)).$$

□

The Toeplitz hyperdeterminant has an integral expression.

LEMMA 3.3. *For any $f \in L^1(\mathbb{T})$, we have*

$$D_n^{[2m]}(f) = \frac{1}{n!} \int_{\mathbb{T}^n} \prod_{j=1}^n f(z_j) \cdot |V(z_1, \dots, z_n)|^{2m} \mathrm{d}z_1 \cdots \mathrm{d}z_n.$$

PROOF. It is straightforward. In fact, since

$$|V(z_1, \dots, z_n)|^2 = \det(z_j^{n-i})_{1 \le i,j \le n} \cdot \det(z_j^{-n+i})_{1 \le i,j \le n},$$

we see that

$$\begin{aligned} &\int_{\mathbb{T}^n} \prod_{j=1}^n f(z_j) \cdot |V(z_1, \dots, z_n)|^{2m} \mathrm{d}z_1 \cdots \mathrm{d}z_n \\ = &\sum_{\sigma_1, \dots, \sigma_{2m} \in \mathfrak{S}_n} \mathrm{sgn}(\sigma_1) \cdots \mathrm{sgn}(\sigma_{2m}) \int_{\mathbb{T}^n} \prod_{j=1}^n f(z_j) z_j^{-\sigma_1(j) - \cdots - \sigma_m(j) + \sigma_{m+1}(j) + \cdots + \sigma_{2m}(j)} \mathrm{d}z_1 \cdots \mathrm{d}z_n. \end{aligned}$$

The last integral is calculated as

$$\prod_{j=1}^n \int_{\mathbb{T}} f(z) z^{-\sigma_1(j) - \cdots - \sigma_m(j) + \sigma_{m+1}(j) + \cdots + \sigma_{2m}(j)} \mathrm{d}z = \prod_{j=1}^n d(\sigma_1(j) + \cdots + \sigma_m(j) - \sigma_{m+1}(j) - \cdots - \sigma_{2m}(j)).$$

Therefore the claim follows from the definition of the Toeplitz hyperdeterminant. □

EXAMPLE 3.4. Let $\mathbf{1}$ be the trivial function $\mathbf{1}(z) = 1$. From Lemma 3.3 and (2.3), we have

$$D_n^{[2m]}(\mathbf{1}) = I_n(1/m) = \frac{(mn)!}{n!\,(m!)^n}.$$

□

3.2. Proof of Main Theorem. The equality (1.3) follows immediately from (1.2) and the duality theorem [**13**, §VI, (10.17)] for Jack functions $P_\lambda^{(\alpha)} \leftrightarrow Q_{\lambda'}^{(1/\alpha)}$, and so we will prove (1.2).

We normalize the Toeplitz hyperdeterminant as

$$\widehat{D}_n^{[2m]}(f) = \frac{D_n^{[2m]}(f)}{D_n^{[2m]}(\mathbf{1})} = \frac{n!\,(m!)^n}{(mn)!} \cdot D_n^{[2m]}(f).$$

We also consider a shifted Toeplitz hyperdeterminant

$$\widehat{D}_{n;a}^{[2m}(f) = \widehat{D}_n^{[2m]}(z^{-a} f(z)), \qquad \text{for any } a \in \mathbb{Z}.$$

Then the expression (1.2) is rewritten as

THEOREM 3.5. $Q_{(L^n)}^{(1/m)}(\mathbf{x}) = \widehat{D}_{n;L}^{[2m]}(G_{\mathbf{x}}^{(1/m)})$.

PROOF. We may assume $G_{\mathbf{x}}^{(1/m)}(z)$ belongs to $L^1(\mathbb{T})$. Apply Lemma 3.3 to $f(z) = z^{-L} G_{\mathbf{x}}^{(1/m)}(z)$. Then

$$\widehat{D}_{n;L}^{[2m]}(G_{\mathbf{x}}^{(1/m)}) = \frac{1}{n!\,I_n(1/m)} \int_{\mathbb{T}^n} \prod_{j=1}^n \left\{ z_j^{-L} \prod_{i=1}^\infty (1 - \mathbf{x}_i z_j)^{-m} \right\} \cdot |V(z_1, \dots, z_n)|^{2m} \mathrm{d}z_1 \cdots \mathrm{d}z_n.$$

Observe

$$P_{(L^n)}^{(\alpha)}(z_1, \dots, z_n) = (z_1 \cdots z_n)^L.$$

(This follows from the reduction identity

$$P^{(\alpha)}_{(\lambda_1,\dots,\lambda_n)}(\mathbf{x}_1,\dots,\mathbf{x}_n) = \mathbf{x}_1\cdots\mathbf{x}_n P^{(\alpha)}_{(\lambda_1-1,\dots,\lambda_n-1)}(\mathbf{x}_1,\dots,\mathbf{x}_n) \qquad \text{if } \lambda_n \neq 0,$$

see [**13**, §VI (4.17)].) From this and the Cauchy identity (2.1), we see that

$$\begin{aligned}\widehat{D}^{[2m]}_{n;L}(G^{(1/m)}_{\mathbf{x}}) = &\sum_{\lambda} P^{(1/m)}_{\lambda}(\mathbf{x}) \frac{1}{n!\, I_n(1/m)} \\ &\times \int_{\mathbb{T}^n} Q^{(1/m)}_{\lambda}(z_1,\dots,z_n)\overline{P^{(1/m)}_{(L^n)}(z_1,\dots,z_n)}\cdot |V(z_1,\dots,z_n)|^{2m} \mathrm{d}z_1\cdots \mathrm{d}z_n.\end{aligned}$$

Therefore we obtain

$$\widehat{D}^{[2m]}_{n;L}(G^{(1/m)}_{\mathbf{x}}) = P^{(1/m)}_{(L^n)}(\mathbf{x}) \prod_{i=1}^{n}\prod_{j=1}^{L} \frac{n+(j-1)/m-i+1}{n+j/m-i} = P^{(1/m)}_{(L^n)}(\mathbf{x}) b^{(1/m)}_{(L^n)} = Q^{(1/m)}_{(L^n)}(\mathbf{x}),$$

by the orthogonality property (2.2). □

Thus, we have proved Theorem 1.1.

3.3. Remarks on hyperdeterminant expressions. From hyperdeterminant expressions (1.2) of Jack functions, we have the following explicit expansion.

PROPOSITION 3.1. *For any positive integers m and L, we have*

$$Q^{(1/m)}_{(L,L)} = (g^{(1/m)}_L)^2 + 2(m!)^2 \sum_{k=1}^{\min\{m,L\}} \frac{(-1)^k}{(m+k)!\,(m-k)!} g^{(1/m)}_{L+k} g^{(1/m)}_{L-k}, \tag{3.1}$$

PROOF. By Theorem 1.1, we have

$$Q^{(1/m)}_{(L,L)} = \frac{(m!)^2}{(2m)!} \sum_{k=0}^{m} c_{m,k} g^{(1/m)}_{L+k} g^{(1/m)}_{L-k}$$

with

$$c_{m,k} = \sum_{\substack{\sigma_1,\dots,\sigma_{2m}\in\mathfrak{S}_2,\\ \sigma_1(1)+\cdots+\sigma_m(1)-\sigma_{m+1}(1)-\cdots-\sigma_{2m}(1)=\ k \text{ or } -k}} \operatorname{sgn}(\sigma_1\cdots\sigma_{2m}).$$

We evaluate $c_{m,k}$. It is clear that $c_{m,0} = 2D^{[2m]}_2(\mathbf{1}) = (2m)!/(m!)^2$. For $k \geq 1$,

$$\begin{aligned} c_{m,k} =& 2 \sum_{\substack{\sigma_1,\dots,\sigma_{2m}\in\mathfrak{S}_2,\\ \sigma_1(1)+\cdots+\sigma_m(1)=\sigma_{m+1}(1)+\cdots+\sigma_{2m}(1)+k}} \operatorname{sgn}(\sigma_1\cdots\sigma_{2m}) \\ =& 2\sum_{i=0}^{m-k} \sum_{\substack{\sigma_1,\dots,\sigma_m\in\mathfrak{S}_2,\\ \sigma_1(1)+\cdots+\sigma_m(1)=2i+m-i+k}} \operatorname{sgn}(\sigma_1\cdots\sigma_m)\cdot \sum_{\substack{\sigma_{m+1},\dots,\sigma_{2m}\in\mathfrak{S}_2,\\ \sigma_{m+1}(1)+\cdots+\sigma_{2m}(1)=2i+m-i}} \operatorname{sgn}(\sigma_{m+1}\cdots\sigma_{2m}). \end{aligned}$$

Here we regard each i as the number of the set $\{j \mid m+1 \leq j \leq 2m,\ \sigma_j(1)=2\}$. Thus,

$$c_{m,k} = 2\sum_{i=0}^{m-k} \binom{m}{i+k}(-1)^{i+k}\cdot\binom{m}{i}(-1)^i = 2(-1)^k \sum_{i=0}^{m-k}\binom{m}{i+k}\binom{m}{i} = 2(-1)^k\binom{2m}{m-k}.$$

Therefore we obtain the claim. □

The expansion (3.1) is a special case of results in [**7**].

4. Hyperpfaffian expression of Jack functions

We express rectangular-shape Jack functions with $\alpha = 2m$ or $1/(2m)$ by a "hyperpfaffian". In particular, if $\alpha = 2$ or $1/2$, the Jack function is expressed by the (ordinary) pfaffian.

4.1. Hyperpfaffians. Let $B = (B(i_1,\dots,i_{2m}))_{[2n]}$ be the array such that

$$B(i_{\tau_1(1)}, i_{\tau_1(2)}, \dots, i_{\tau_m(2m-1)}, i_{\tau_m(2m)}) = \mathrm{sgn}(\tau_1)\cdots\mathrm{sgn}(\tau_m) B(i_1,\dots,i_{2m}) \tag{4.1}$$

for any $(\tau_1,\dots,\tau_m) \in (\mathfrak{S}_2)^m$. Here each $\tau_s \in \mathfrak{S}_2$ permutates $2s-1$ with $2s$. For example, if $m=2$, relation (4.1) is described as

$$B(i_1,i_2,i_3,i_4) = -B(i_2,i_1,i_3,i_4) = -B(i_1,i_2,i_4,i_3) = B(i_2,i_1,i_4,i_3).$$

DEFINITION 4.1. For an array $B = (B(i_1,\dots,i_{2m}))_{[2n]}$ satisfying (4.1), we define the *hyperpfaffian* of B by

$$\mathrm{pf}^{[2m]}(B) := \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_m \in \mathfrak{E}_{2n}} \mathrm{sgn}(\sigma_1\cdots\sigma_m) \prod_{i=1}^{n} B(\sigma_1(2i-1), \sigma_1(2i), \dots, \sigma_m(2i-1), \sigma_m(2i)), \tag{4.2}$$

where $\mathfrak{E}_{2n} := \{\sigma \in \mathfrak{S}_{2n} \mid \sigma(2i-1) < \sigma(2i)\ (1 \le i \le n)\}$. □

When $m=1$, it is just the ordinary pfaffian $\mathrm{pf}(B)$ of an alternating matrix $B = (B(i,j))_{1 \le i,j \le 2n}$.

4.2. Relations with Barvinok's hyperpfaffians. Our hyperpfaffian is seen to be a special case of Barvinok's hyperpfaffian [**2**]. We give the explicit relationship between these two hyperpfaffians. Let $M = (M(i_1,\dots,i_{2m}))_{[2mn]}$ be an array satisfying

$$M(i_{\tau(1)},\dots,i_{\tau(2m)}) = \mathrm{sgn}(\tau) M(i_1,\dots,i_{2m})$$

for any $\tau \in \mathfrak{S}_{2m}$. Barvinok [**2**] (see also [**11**]) defines his hyperpfaffian by

$$\mathrm{Pf}^{[2m]}(M) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{E}_{2mn,2m}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} M(\sigma(2m(i-1)+1), \sigma(2m(i-1)+2), \dots, \sigma(2mi)),$$

where

$$\mathfrak{E}_{2mn,2m} = \{\sigma \in \mathfrak{S}_{2mn} \mid \sigma(2m(i-1)+1) < \sigma(2m(i-1)+2) < \cdots < \sigma(2mi)\ (1 \le i \le n)\}.$$

As the following proposition states, our hyperpfaffian $\mathrm{pf}^{[2m]}$ is a special case of $\mathrm{Pf}^{[2m]}$. However, our hyperpfaffian is more useful than Barvinok's when we state our theorems below.

PROPOSITION 4.1. *Let $B = (B(i_1,\dots,i_{2m}))_{[2n]}$ be an array satisfying* (4.1). *Let*

$$M = (M(i_1,\dots,i_{2m}))_{[2mn]}$$

be the array whose entries $M(i_1,\dots,i_{2m})$ are given as follows if $i_1 < \cdots < i_{2m}$. If there exist $1 \le r_1,\dots,r_{2m} \le 2n$ such that $i_{2s-1} = 2n(s-1)+r_{2s-1}$ and $i_{2s} = 2n(s-1)+r_{2s}$ for any $1 \le s \le m$, then $M(i_1,\dots,i_{2m}) = B(r_1,\dots,r_{2m})$. Otherwise define $M(i_1,\dots,i_{2m}) = 0$. Then we have

$$\mathrm{pf}^{[2m]}(B) = \mathrm{Pf}^{[2m]}(M).$$

PROOF. The value $\prod_{i=1}^{n} M(\sigma(2m(i-1)+1), \sigma(2m(i-1)+2), \dots, \sigma(2mi))$ is zero unless the permutation $\sigma \in \mathfrak{S}_{2mn}$ satisfies

$$2n(s-1)+1 \le \ \sigma(2m(i-1)+2s-1), \sigma(2m(i-1)+2s)\ \le 2ns$$

for any $1 \le i \le n$ and $1 \le s \le m$. Therefore we have

$$\mathrm{Pf}^{[2m]}(M) = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_m \in \mathfrak{E}_{2n}} \epsilon(\sigma_1,\dots,\sigma_m) \prod_{i=1}^{n} M(\cdots, 2n(s-1)+\sigma_s(2i-1), 2n(s-1)+\sigma_s(2i), \cdots),$$

where $\epsilon(\sigma_1,\dots,\sigma_m)$ is the signature of the permutation σ defined by

$$\sigma(2m(i-1)+2s-1) = 2n(s-1)+\sigma_s(2i-1) \qquad \text{and} \qquad \sigma(2m(i-1)+2s) = 2n(s-1)+\sigma_s(2i)$$

for any $1 \le i \le n$ and $1 \le s \le m$. Hence we have $\mathrm{Pf}^{[2m]}(M) = \epsilon(\mathrm{id},\dots,\mathrm{id})\mathrm{pf}^{[2m]}(B)$, and so $\epsilon(\mathrm{id},\dots,\mathrm{id}) = \mathrm{sgn}(\rho)$, where ρ is the permutation such that

$$\rho(2m(i-1)+2s-1) = 2n(s-1)+2i-1 \qquad \text{and} \qquad \rho(2m(i-1)+2s) = 2n(s-1)+2i.$$

Now it is straightforward to see $\mathrm{sgn}(\rho) = 1$. □

4.3. Proof of Theorem 1.2. The following lemma gives the connection between the Toeplitz hyperdeterminant and hyperpfaffian.

LEMMA 4.2. *For a function $f(z)=\sum_{k\in\mathbb{Z}} d(k)z^k \in L^1(\mathbb{T})$, we have*

$$D_n^{[4m]}(f)=\mathrm{pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot d((2n+1)m-\sum_{k=1}^{2m} i_k)\right)_{[2n]}.$$

PROOF. From Lemma 3.3, it is sufficient to prove the equality

$$\mathrm{pf}^{[2m]}\left(R(i_1,\dots,i_{2m})\right)_{[2n]}=\frac{1}{n!}\int_{\mathbb{T}^n} f(z_1)f(z_2)\cdots f(z_n)\cdot\prod_{1\le j<k\le n}|z_j-z_k|^{4m}\mathrm{d}z_1\cdots\mathrm{d}z_n, \tag{4.3}$$

where $R(i_1,\dots,i_{2m})=\prod_{s=1}^m(i_{2s}-i_{2s-1})\cdot d((2n+1)m-i_1-i_2-\cdots-i_{2m})$.

First we recall the formula (see [**15**, pp.216] for example)

$$\det\left(z_k^{j-n-\frac{1}{2}}\mid (j-n-\frac{1}{2})z_k^{j-n-\frac{1}{2}}\right)_{1\le j\le 2n,1\le k\le n}=\prod_{1\le j<k\le n}|z_j-z_k|^4.$$

Here $\det(a_{j,k}|b_{j,k})_{1\le j\le 2n,1\le k\le n}$ denotes the determinant of the matrix whose j-th row is given by

$$(a_{j,1}\ b_{j,1}\ a_{j,2}\ b_{j,2}\ \cdots\ a_{j,n}\ b_{j,n}).$$

From the determinantal expansion

$$\det(a_{i,j})_{1\le i,j\le 2n}=\sum_{\sigma\in\mathfrak{E}_{2n}}\mathrm{sgn}(\sigma)\prod_{j=1}^{n}(a_{\sigma(2j-1),2j-1}a_{\sigma(2j),2j}-a_{\sigma(2j-1),2j}a_{\sigma(2j),2j-1}), \tag{4.4}$$

we have

$$\prod_{1\le j<k\le n}|z_j-z_k|^4=\sum_{\sigma\in\mathfrak{E}_{2n}}\mathrm{sgn}(\sigma)\prod_{j=1}^{n}(\sigma(2j)-\sigma(2j-1))z_j^{\sigma(2j-1)+\sigma(2j)-2n-1}.$$

Therefore we see that

$$\begin{aligned}
&\frac{1}{n!}\int_{\mathbb{T}^n} f(z_1)f(z_2)\cdots f(z_n)\cdot\prod_{1\le j<k\le n}|z_j-z_k|^{4m}\mathrm{d}z_1\cdots\mathrm{d}z_n\\
=&\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{E}_{2n}}\mathrm{sgn}(\sigma_1)\cdots\mathrm{sgn}(\sigma_m)\prod_{j=1}^{n}\left(\prod_{s=1}^{m}(\sigma_s(2j)-\sigma_s(2j-1))\int_{\mathbb{T}} f(z)\prod_{s=1}^{m} z^{\sigma_s(2j-1)+\sigma_s(2j)-2n-1}\mathrm{d}z\right)\\
=&\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{E}_{2n}}\mathrm{sgn}(\sigma_1)\cdots\mathrm{sgn}(\sigma_m)\prod_{j=1}^{n}\left(\prod_{s=1}^{m}(\sigma_s(2j)-\sigma_s(2j-1))\cdot d(m(2n+1)-\sum_{s=1}^{m}(\sigma_s(2j-1)+\sigma_s(2j)))\right)\\
=&\mathrm{pf}^{[2m]}(R(i_1,\dots,i_{2m}))_{[2n]}.
\end{aligned}$$

□

We are interested in a direct algebraic proof of Lemma 4.2. Note that, when $m=1$, any Toeplitz hyperdeterminant is a pfaffian:

$$D_n^{[4]}(f)=\mathrm{pf}((j-i)d(2n+1-i-j))_{[2n]}.$$

From Theorem 1.1 and Lemma 4.2, we obtain the following.

THEOREM 4.3. *Let m,n,L be positive integers. Then we have*

$$\begin{aligned}
Q_{(L^n)}^{(1/(2m))}=&\frac{n!\,\{(2m)!\}^n}{(2mn)!}\cdot\mathrm{pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot g_{L+(2n+1)m-\sum_{k=1}^{2m}i_k}^{(1/(2m))}\right)_{[2n]},\\
P_{(n^L)}^{(2m)}=&\frac{n!\,\{(2m)!\}^n}{(2mn)!}\cdot\mathrm{pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot e_{L+(2n+1)m-\sum_{k=1}^{2m}i_k}\right)_{[2n]}.
\end{aligned}$$

□

Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be a rectangular partition of length $\le n$, i.e, we can write as $\lambda=(L,L,\dots,L,0,0,\dots,0)$. If $\lambda_n\neq 0$ (i.e., $\lambda_n=L$) then Theorem 1.2 is the case $m=1$ of Theorem 4.3. To complete the proof of Theorem 1.2, we prove that, if $\lambda_n=0$, it holds

$$\mathrm{pf}((j-i)g_{\gamma_j+2n+1-i-j})_{1\le i<j\le 2n}=(2n-1)\mathrm{pf}((j-i)g_{\widetilde{\gamma}_j+2n-1-i-j})_{1\le i<j\le 2n-2}, \tag{4.5}$$

where γ_j are given by (1.6) and $\widetilde{\gamma}_j$ are given by the same way for $(\lambda_1,\dots,\lambda_{n-1})$, i.e.,

$$(\gamma_1,\dots,\gamma_{2n-2})=(\lambda_{n-1},\dots,\lambda_2,\lambda_1,\lambda_1,\lambda_2,\dots,\lambda_{n-1}).$$

We can check the equation (4.5) immediately if we expand the pfaffian on its left hand side with respect to the $2n$-th column.

5. Summation identities for Schur functions

Finally, we obtain some summation identities for Schur functions involving hyperdeterminants and hyperpfaffians.

The Schur function is the Jack function with parameter $\alpha=1$. For a partition λ of length $\le n$, the Schur function in n variables $s_\lambda(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is given by the quotient

$$s_\lambda(\mathbf{x}_1,\dots,\mathbf{x}_n)=\frac{\det(\mathbf{x}_j^{\lambda_i+n-i})_{1\le i,j\le n}}{V(\mathbf{x}_1,\dots,\mathbf{x}_n)},$$

where $V(\mathbf{x}_1,\dots,\mathbf{x}_n)=\prod_{1\le i<j\le n}(\mathbf{x}_i-\mathbf{x}_j)$.

The Schur functions have the following well-known summation formulas [**13**, §I-4, Ex. 6], [**6**]:

$$\sum_{\lambda:\ell(\lambda)\le n}s_\lambda(\mathbf{x})s_\lambda(\mathbf{y})=\frac{1}{V(\mathbf{x})V(\mathbf{y})}\cdot\det\left(\frac{1}{1-\mathbf{x}_i\mathbf{y}_j}\right)_{1\le i,j\le n}, \tag{5.1}$$

$$\sum_{\lambda:\ell(\lambda)\le n}s_\lambda(\mathbf{x})=\frac{1}{V(\mathbf{x})}\cdot\mathrm{pf}\left(\frac{\mathbf{x}_i-\mathbf{x}_j}{(1-\mathbf{x}_i)(1-\mathbf{x}_j)(1-\mathbf{x}_i\mathbf{x}_j)}\right)_{1\le i,j\le n}, \tag{5.2}$$

where $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)$ and $\mathbf{y}=(\mathbf{y}_1,\dots,\mathbf{y}_n)$. Here we assume n is even in expression (5.2). Note that the determinant on the right-hand side in (5.1) is called Cauchy's determinant. We obtain extensions to higher degree of expressions (5.1) and (5.2).

THEOREM 5.1. *Let $\mathbf{x}^{(i)}=(\mathbf{x}_1^{(i)},\mathbf{x}_2^{(i)},\dots,\mathbf{x}_n^{(i)})$ be the sequence of n variables for each $1\le i\le 2m$. Then*

$$\sum_{\lambda:\ell(\lambda)\le n}s_\lambda(\mathbf{x}^{(1)})s_\lambda(\mathbf{x}^{(2)})\cdots s_\lambda(\mathbf{x}^{(2m)})=\prod_{i=1}^{2m}\frac{1}{V(\mathbf{x}^{(i)})}\det{}^{[2m]}\left(\frac{1}{1-\mathbf{x}_{i_1}^{(1)}\cdots\mathbf{x}_{i_{2m}}^{(2m)}}\right)_{[n]}. \tag{5.3}$$

PROOF. We identify each partition λ of length $\ell(\lambda)\le n$ with the n-element set $\{k_i:=\lambda_i+n-i\mid i=1,2,\dots,n\}$ of non-negative integers. Then we have

$$\begin{aligned}
&\prod_{i=1}^{2m}V(\mathbf{x}^{(i)})\cdot\sum_{\lambda:\ell(\lambda)\le n}s_\lambda(\mathbf{x}^{(1)})s_\lambda(\mathbf{x}^{(2)})\cdots s_\lambda(\mathbf{x}^{(2m)})\\
=&\sum_{\infty>k_1>\cdots>k_n\ge 0}\prod_{i=1}^{2m}\det((\mathbf{x}_p^{(i)})^{k_q})_{1\le p,q\le n}=\frac{1}{n!}\sum_{k_1,\dots,k_n=0}^{\infty}\prod_{i=1}^{2m}\left(\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\prod_{p=1}^{n}(\mathbf{x}_{\sigma(p)}^{(i)})^{k_p}\right)\\
=&\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_{2m}\in\mathfrak{S}_n}\mathrm{sgn}(\sigma_1\cdots\sigma_{2m})\prod_{p=1}^{n}\left(\sum_{k=0}^{\infty}\prod_{i=1}^{2m}(\mathbf{x}_{\sigma_i(p)}^{(i)})^{k}\right)=\det{}^{[2m]}\left((1-\prod_{k=1}^{2m}\mathbf{x}_{i_k}^{(k)})^{-1}\right)_{[n]}.
\end{aligned}$$

□

REMARK 5.2. Cauchy's determinant can be given as a product

$$\det\left(\frac{1}{1-\mathbf{x}_i\mathbf{y}_j}\right)_{1\le i,j\le n}=V(\mathbf{x})V(\mathbf{y})\prod_{i,j=1}^{n}\frac{1}{1-\mathbf{x}_i\mathbf{y}_j}.$$

But we can not expect such a product expression for the hyperdeterminant

$$\det^{[2m]}\left(\frac{1}{1-\mathbf{x}^{(1)}_{i_1}\cdots\mathbf{x}^{(2m)}_{i_{2m}}}\right)_{[n]}.$$

□

One may see expression (5.3) as a simple multi-version of (5.1). An odd-product analogue of (5.3) is given as follows.

Theorem 5.3. *Let m be an odd positive number and $\mathbf{x}^{(i)}=(\mathbf{x}^{(i)}_1,\mathbf{x}^{(i)}_2,\dots,\mathbf{x}^{(i)}_{2n})$ be the sequence of $2n$ variables for each $1\le i\le m$. Then we have*

$$\sum_{\lambda:\ell(\lambda)\le 2n} s_\lambda(\mathbf{x}^{(1)})s_\lambda(\mathbf{x}^{(2)})\cdots s_\lambda(\mathbf{x}^{(m)})=\prod_{i=1}^{m}\frac{1}{V(\mathbf{x}^{(i)})}\cdot \operatorname{pf}^{[2m]}\left(\frac{\sum_{p=0}^{\infty}\prod_{s=1}^{m}\{(\mathbf{x}^{(s)}_{i_{2s-1}})^{p+1}-(\mathbf{x}^{(s)}_{i_{2s}})^{p+1}\}}{1-\mathbf{x}^{(1)}_{i_1}\mathbf{x}^{(1)}_{i_2}\cdots\mathbf{x}^{(m)}_{i_{2m-1}}\mathbf{x}^{(m)}_{i_{2m}}}\right)_{[2n]}.$$

Proof. A similar way in the proof of Theorem 5.1 gives

$$\begin{aligned}
&\prod_{i=1}^{2m} V(\mathbf{x}^{(i)})\cdot\sum_{\lambda:\ell(\lambda)\le 2n} s_\lambda(\mathbf{x}^{(1)})s_\lambda(\mathbf{x}^{(2)})\cdots s_\lambda(\mathbf{x}^{(m)})\\
&=\sum_{\infty>k_1>\cdots>k_{2n}\ge 0}\operatorname{pf}(\operatorname{sgn}(k_p-k_q))_{1\le p,q\le 2n}\prod_{s=1}^{m}\det((\mathbf{x}^{(s)}_p)^{k_q})_{1\le p,q\le 2n}\\
&=\frac{1}{(2n)!\,n!}\sum_{\sigma\in\mathfrak{E}_{2n}}\operatorname{sgn}(\sigma)\sum_{k_1,\dots,k_{2n}=0}^{\infty}\prod_{j=1}^{n}\operatorname{sgn}(k_{\sigma(2j-1)}-k_{\sigma(2j)})\cdot\prod_{s=1}^{m}\det((\mathbf{x}^{(s)}_p)^{k_q})_{1\le p,q\le 2n},
\end{aligned}$$

where $\operatorname{sgn}(x)=x/|x|$ for $x\ne 0$ and $\operatorname{sgn}(0)=0$. Since m is odd, it equals

$$\begin{aligned}
&=\frac{1}{(2n)!\,n!}\sum_{\sigma\in\mathfrak{E}_{2n}}\sum_{k_1,\dots,k_{2n}=0}^{\infty}\prod_{j=1}^{n}\operatorname{sgn}(k_{\sigma(2j-1)}-k_{\sigma(2j)})\prod_{s=1}^{m}\det((\mathbf{x}^{(s)}_p)^{k_{\sigma(q)}})_{1\le p,q\le 2n}\\
&=\frac{1}{(2n)!\,n!}(\#\mathfrak{E}_{2n})\sum_{k_1,\dots,k_{2n}=0}^{\infty}\prod_{j=1}^{n}\operatorname{sgn}(k_{2j-1}-k_{2j})\prod_{s=1}^{m}\det((\mathbf{x}^{(s)}_p)^{k_q})_{1\le p,q\le 2n}\\
&=\frac{1}{n!\,2^n}\sum_{k_1,\dots,k_{2n}=0}^{\infty}\prod_{j=1}^{n}\operatorname{sgn}(k_{2j-1}-k_{2j})\prod_{s=1}^{m}\det((\mathbf{x}^{(s)}_p)^{k_q})_{1\le p,q\le 2n}.
\end{aligned}$$

By expression (4.4), we see that

$$\begin{aligned}
&=\frac{1}{n!\,2^n}\sum_{k_1,\dots,k_{2n}=0}^{\infty}\prod_{j=1}^{n}\operatorname{sgn}(k_{2j-1}-k_{2j})\\
&\quad\times\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{E}_{2n}}\operatorname{sgn}(\sigma_1\cdots\sigma_m)\prod_{s=1}^{m}\prod_{j=1}^{n}\{(\mathbf{x}^{(s)}_{\sigma_s(2j-1)})^{k_{2j-1}}(\mathbf{x}^{(s)}_{\sigma_s(2j)})^{k_{2j}}-(\mathbf{x}^{(s)}_{\sigma_s(2j-1)})^{k_{2j}}(\mathbf{x}^{(s)}_{\sigma_s(2j)})^{k_{2j-1}}\}\\
&=\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{E}_{2n}}\operatorname{sgn}(\sigma_1\cdots\sigma_m)\\
&\quad\times\prod_{j=1}^{n}\left[\frac{1}{2}\sum_{k,l=0}^{\infty}\operatorname{sgn}(k-l)\prod_{s=1}^{m}\{(\mathbf{x}^{(s)}_{\sigma_s(2j-1)})^{k}(\mathbf{x}^{(s)}_{\sigma_s(2j)})^{l}-(\mathbf{x}^{(s)}_{\sigma_s(2j-1)})^{l}(\mathbf{x}^{(s)}_{\sigma_s(2j)})^{k}\}\right].
\end{aligned}$$

Here a simple calculation gives

$$[\quad]\text{ on the last equality}=\frac{\sum_{p=0}^{\infty}\prod_{s=1}^{m}\{(\mathbf{x}^{(s)}_{\sigma_s(2j-1)})^{p+1}-(\mathbf{x}^{(s)}_{\sigma_s(2j)})^{p+1}\}}{1-\mathbf{x}^{(1)}_{\sigma_1(2j-1)}\mathbf{x}^{(1)}_{\sigma_1(2j)}\cdots\mathbf{x}^{(m)}_{\sigma_m(2j-1)}\mathbf{x}^{(m)}_{\sigma_m(2j)}}.$$

Therefore our claim follows. □

Furthermore, using the Jacobi-Trudi formula $s_\lambda = \det(h_{\lambda_i - i + j})_{i,j \geq 1}$, we can also obtain the following summation formulas in a similar way to the last theorems.

THEOREM 5.4. *Let $\mathbf{x}^{(i)} = (\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(i)}, \dots)$ for each $i \geq 1$. Then, for each positive integer m, we have*

$$\sum_{\lambda:\ell(\lambda)\leq n} s_\lambda(\mathbf{x}^{(1)}) \cdots s_\lambda(\mathbf{x}^{(2m)}) = \det{}^{[2m]} \left(\sum_{k \geq 0} h_{k-i_1}(\mathbf{x}^{(1)}) h_{k-i_2}(\mathbf{x}^{(2)}) \cdots h_{k-i_{2m}}(\mathbf{x}^{(2m)}) \right)_{[n]} . \tag{5.4}$$

Also, for an odd positive integer m,

$$\sum_{\lambda:\ell(\lambda)\leq 2n} s_\lambda(\mathbf{x}^{(1)}) \cdots s_\lambda(\mathbf{x}^{(m)}) = \mathrm{pf}^{[2m]} \left(\sum_{k,l \geq 0} \prod_{s=1}^{m} \det \begin{pmatrix} h_{k+l+1-i_{2s-1}}(\mathbf{x}^{(s)}) & h_{l-i_{2s-1}}(\mathbf{x}^{(s)}) \\ h_{k+l+1-i_{2s}}(\mathbf{x}^{(s)}) & h_{l-i_{2s}}(\mathbf{x}^{(s)}) \end{pmatrix} \right)_{[2n]} . \tag{5.5}$$

□

The cases $m = 1$ in expressions (5.4) and (5.5) are seen in [**4**] and [**17**] respectively.

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GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, JAPAN
E-mail address: `shom@math.kyushu-u.ac.jp`