Real Wishart matrices and Haar-distributed orthogonal matrices

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Monday, September 6th, 2010

What is a Wishart matrix ?

$\mathrm{Sym}(d)$: the set of all $d \times d$ real symmetric matrices.
$\mathrm{Sym}^+(d)$: the set of all **positive definite** matrices in $\mathrm{Sym}(d)$.

A classical definition for a real Wishart matrix

Let $\sigma \in \mathrm{Sym}^+(d)$. Let $X_1, X_2, \dots, X_p$ be i.i.d. $\mathbb{R}^d$-valued random column vectors $\sim \mathrm{N}_d(\mathbf{0}, \frac{1}{2}\sigma)$. Then the random matrix

$$W = X_1 X_1^t + X_2 X_2^t + \cdots + X_p X_p^t$$

in $\mathrm{Sym}^+(d)$ is called a real **Wishart matrix** with parameters (p, σ).

Its Laplace transform (or moment-generating function):

$$\mathbb{E}[e^{\mathrm{tr}(\theta W)}] = \det(I - \theta\sigma)^{-p/2}$$

for $\theta \in \mathrm{Sym}(d)$ such that $\sigma^{-1} - \theta \in \mathrm{Sym}^+(d)$.

What is a Wishart matrix ?

Definition for a real Wishart matrix

Let $\sigma \in \mathrm{Sym}^+(d)$. Suppose

$$\beta \in \left\{ \frac{1}{2}, \frac{2}{2}, \dots, \frac{d-1}{2} \right\} \cup \left(\frac{d-1}{2}, +\infty \right). \qquad \text{(G)}$$

Define a random matrix W in $\mathrm{Sym}^+(d)$ by the Laplace transform

$$\mathbb{E}[e^{\mathrm{tr}(\theta W)}] = \det(I - \theta\sigma)^{-\beta}. \qquad (*)$$

It is called a real Wishart matrix. We write as $W \sim \mathrm{W}_d(\beta, \sigma)$.

For $\beta > 0$ and $\sigma \in \mathrm{Sym}^+(d)$, there exists a random matrix W satisfying $(*)$ **if and only if** (G) holds true. [Gindikin (1975)]

Density function

If $\beta > \frac{d-1}{2}$, the Wishart matrix $W \sim \mathrm{W}_d(\beta, \sigma)$ has a density function with respect to the Lebesgue measure on $\mathrm{Sym}(d)$.

$$\Gamma_d(\beta)^{-1} (\det \sigma)^{-\beta} (\det w)^{\beta - \frac{d+1}{2}} e^{-\mathrm{tr}(\sigma^{-1} w)} \qquad (w \in \mathrm{Sym}^+(d))$$

where

$$\Gamma_d(\beta) = \pi^{d(d-1)/4} \prod_{j=1}^{d} \Gamma\left(\beta - \frac{j-1}{2}\right).$$

When $d = 1$, the Wishart matrix is a gamma distribution.

Main problem

Let $W = (W_{ij})_{1 \le i,j \le d} \sim \mathrm{W}_d(\beta, \sigma)$ and let $W^{-1} = (W^{ij})_{1 \le i,j \le d}$ be its inverse.

Compute **general moments** of the forms

$$\mathbb{E}[W_{i_1 j_1} W_{i_2 j_2} \cdots W_{i_n j_n}] \qquad \text{(for } W\text{)}$$

and

$$\mathbb{E}[W^{i_1 j_1} W^{i_2 j_2} \cdots W^{i_n j_n}] \qquad \text{(for } W^{-1}\text{)}.$$

Main result: to give an exact formula for $\mathbb{E}[W^{i_1 j_1} W^{i_2 j_2} \cdots W^{i_n j_n}]$

History

- A formula for the moments for a **complex** Wishart matrix is given by [Graczyk, Letac, Massam (2003)].
- For a **real** Wishart matrix W, [GLM (2005)] computed $\mathbb{E}[W_{i_1 j_1} W_{i_2 j_2} \cdots W_{i_n j_n}]$.
- However, they could not compute $\mathbb{E}[W^{i_1 j_1} W^{i_2 j_2} \cdots W^{i_n j_n}]$.

Related works for other random matrices:

- For **Gaussian Unitary Ensemble (GUE)** matrix $H = (H_{ij})$, the calculus for the moment $\mathbb{E}[H_{i_1 j_1} H_{i_2 j_2} \cdots H_{i_n j_n}]$ is well known. — Wick formula.

History

Furthermore

- For a **Haar-distributed unitary** matrix $U = (U_{ij}) \in \mathrm{U}(N)$, moments of the form

$$\mathbb{E}[U_{i_1 j_1} U_{i_2 j_2} \cdots U_{i_n j_n} \overline{U_{k_1 l_1} U_{k_2 l_2} \cdots U_{k_m l_m}}]$$

  are studied by [Collins (2003)], [M, Novak (2009)] and so on.
- For a **Haar-distributed orthogonal** matrix $O = (O_{ij}) \in \mathrm{O}(N)$, moments of the form $\mathbb{E}[O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_n j_n}]$ are studied by [Collins, Śniady (2006)], [Collins, M (2009)] and so on.

Perfect matchings

Let $\mathcal{M}(2n)$ be the subset of S_{2n} given by

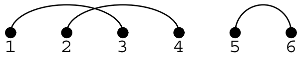
$$\mathcal{M}(2n) = \left\{ \mathfrak{m} = \begin{pmatrix} 1 & 2 & \dots & \dots & 2n-1 & 2n \\ \mathfrak{m}(1) & \mathfrak{m}(2) & \dots & \dots & \mathfrak{m}(2n-1) & \mathfrak{m}(2n) \end{pmatrix} \in S_{2n} \;\middle|\; \begin{array}{l} \mathfrak{m}(2i-1) < \mathfrak{m}(2i) \ (1 \le i \le n), \\ \mathfrak{m}(1) < \mathfrak{m}(3) < \cdots < \mathfrak{m}(2n-1) \end{array} \right\}$$

Example. $\mathcal{M}(4)$ consists of three permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

We identify an element in $\mathcal{M}(2n)$ with a perfect matching:

$$\mathcal{M}(6) \ni \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix} \quad \leftrightarrow$$



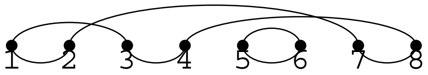
Coset-types for permutations

For each permutation $g \in S_{2n}$, we define the coset-type ρ, which is a partition of n, as follows.

Consider the undirected graph $G(g)$, defined as follows:
vertex set $\{1, 2, \dots, 2n\}$;
edges $\{g(2i-1), g(2i)\}, \{2i-1, 2i\}, \quad i = 1, 2, \dots, n$.

This graph has some loops of even lengths $2\rho_1 \geq 2\rho_2 \geq \cdots$, say.
Thus $g \in S_{2n}$ determines a partition $\rho = (\rho_1, \rho_2, \dots)$ of n.

Example. Let $g = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 6 & 5 & 2 & 7 & 4 & 8 \end{smallmatrix}\right) \in S_8$. Then its coset-type is $(3, 1) \vdash 4$.



Moment $\mathbb{E}[W_{k_1 k_2} W_{k_3 k_4} \cdots W_{k_{2n-1} k_{2n}}]$

Given $g \in S_{2n}$, denote by $\kappa(g)$ the length of the coset-type ρ of g: $\kappa(g) = \ell(\rho)$.

Theorem. [Graczyk, Letac, Massam (2003)]

Let $W \sim \mathrm{W}_d(\beta, \sigma)$. Given indices $k_1, k_2, \dots, k_{2n}$ from $\{1, 2, \dots, d\}$, we have

$$\mathbb{E}[W_{k_1 k_2} W_{k_3 k_4} \cdots W_{k_{2n-1} k_{2n}}] = 2^{-n} \sum_{\mathfrak{m} \in \mathcal{M}(2n)} (2\beta)^{\kappa(\mathfrak{m})} \prod_{\{p,q\} \in \mathfrak{m}} \sigma_{k_p k_q}.$$

Example. $(n = 2)$

$$\mathbb{E}[W_{k_1 k_2} W_{k_3 k_4}] = \beta^2 \sigma_{k_1 k_2} \sigma_{k_3 k_4} + \frac{\beta}{2} \sigma_{k_1 k_3} \sigma_{k_2 k_4} + \frac{\beta}{2} \sigma_{k_1 k_4} \sigma_{k_2 k_3}.$$

Moment $\mathbb{E}[W^{k_1 k_2} W^{k_3 k_4} \dots W^{k_{2n-1} k_{2n}}]$

Main Theorem.

Let $W \sim \mathrm{W}_d(\beta, \sigma)$ and let $W^{-1} = (W^{ij})$ be its inverse matrix. Let $\sigma^{-1} = (\sigma^{ij})$. Put $\gamma := \beta - \frac{d+1}{2}$ and suppose $n < \gamma + 1$. Then, given indices $k_1, k_2, \dots, k_{2n}$ from $\{1, 2, \dots, d\}$, we have

$$\begin{aligned} &\mathbb{E}[W^{k_1 k_2} W^{k_3 k_4} \dots W^{k_{2n-1} k_{2n}}] \\ =&(-1)^n 2^n \sum_{\mathfrak{m} \in \mathcal{M}(2n)} \mathrm{Wg}(\mathfrak{m}; -2\gamma) \prod_{\{p,q\} \in \mathfrak{m}} \sigma^{k_p k_q}. \end{aligned}$$

Here the definition of the function $\mathrm{Wg}(g; z) \in \mathbb{R}$ with $g \in S_{2n}$ and $z \in \mathbb{R}$ will be given later.

It is called the orthogonal Weingarten function.

Example. Let $W \sim \mathrm{W}_d(\beta, \sigma)$. Put $\gamma = \beta - \frac{d+1}{2}$.
Suppose $\gamma > 0$ but $\gamma \neq 1$.

$$\mathbb{E}[W^{k_1 k_2} W^{k_3 k_4}] = \frac{(2\gamma - 1)\sigma^{k_1 k_2}\sigma^{k_3 k_4} + \sigma^{k_1 k_3}\sigma^{k_2 k_4} + \sigma^{k_1 k_4}\sigma^{k_2 k_3}}{\gamma(\gamma - 1)(2\gamma + 1)}.$$

Hyperoctahedral group

Let H_n be the hyperoctahedral group of degree n: the group H_n is the subgroup of S_{2n} generated by

$$(2k-1\ 2k) \quad (1 \le k \le n), \qquad (2i-1\ 2j-1)(2i\ 2j) \quad (1 \le i < j \le n).$$

- $|H_n| = 2^n n!$. Elements of $\mathcal{M}(2n)$ are representatives of the coset S_{2n}/H_n: $S_{2n} = \bigsqcup_{\mathfrak{m} \in \mathcal{M}(2n)} \mathfrak{m} H_n$.
- Given $g, g' \in S_{2n}$, their coset-types coincide if and only if $H_n g H_n = H_n g' H_n$. Hence

$$S_{2n} = \bigsqcup_{\rho \vdash n} H_\rho,$$

where $H_\rho = \{g \in S_{2n} \mid \text{the coset-type of } g \text{ is } \rho\}$.

Zonal spherical functions

Given a partition μ of k, denote by χ^μ the irreducible character of S_k corresponding to μ.
Let f^μ be the degree of χ^μ: $f^\mu = \chi^\mu(\mathrm{id}_{S_k})$.

For each $\lambda \vdash n$, we define the function ω^λ on S_{2n} by

$$\omega^\lambda(g) = \frac{1}{2^n n!} \sum_{\zeta \in H_n} \chi^{2\lambda}(g\zeta) \qquad (g \in S_{2n}),$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$.
The ω^λ are called zonal spherical functions for the Gelfand pair (S_{2n}, H_n).

$$\omega^\lambda(\zeta g) = \omega^\lambda(g\zeta) = \omega^\lambda(g) \qquad (\forall g \in S_{2n},\ \forall \zeta \in H_n).$$

A polynomial

For each partition λ and a number z, we put

$$C_\lambda(z) = \prod_{(i,j)\in\lambda} (z + 2j - i - 1).$$

0	2	4	6
-1	1	3	5
-2	0	2	

where the product runs over all boxes of the Young diagram λ.

Example.

$$\begin{aligned} C_{(4,4,3)}(z) = & z(z+2)(z+4)(z+6) \\ & \times (z-1)(z+1)(z+3)(z+5) \\ & \times (z-2)(z+0)(z+2). \end{aligned}$$

From now, we assume z satisfies $C_\lambda(z) \neq 0$ for all $\lambda \vdash n$.

Definition of $\mathrm{Wg}(g; z)$

Definition [Collins, M (2009)]

$$\mathrm{Wg}(g; z) = \frac{1}{(2n-1)!!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C_\lambda(z)} \omega^\lambda(g) \qquad (g \in S_{2n}).$$

Example. Let $n = 2$ and let $g \in S_4$.
If the coset-type of g is $(1, 1)$,

$$\mathrm{Wg}(g; z) = \frac{1}{3} \left(\frac{1}{z(z+2)} 1 + \frac{2}{z(z-1)} 1 \right) = \frac{z+1}{z(z+2)(z-1)}.$$

If the coset-type of g is (2),

$$\mathrm{Wg}(g; z) = \frac{1}{3} \left(\frac{1}{z(z+2)} 1 + \frac{2}{z(z-1)} \left(-\frac{1}{2} \right) \right) = \frac{-1}{z(z+2)(z-1)}.$$

Haar-distributed orthogonal matrix

Let $\mathrm{O}(N)$ be the Lie group of $N \times N$ real orthogonal matrices, equipped Haar probability measure.

Theorem. [Collins, Śniady (2006)], [Collins, M (2009)]

Let $O = (O_{ij}) \in \mathrm{O}(N)$ be a Haar-distributed orthogonal matrix. Then

$$\mathbb{E}[O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_{2n} j_{2n}}] = \sum_{\mathfrak{m},\mathfrak{n} \in \mathcal{M}(2n)} \mathrm{Wg}(\mathfrak{m}^{-1}\mathfrak{n}; N) \left(\prod_{\{p,q\}\in\mathfrak{m}} \delta_{i_p, i_q} \right) \left(\prod_{\{p,q\}\in\mathfrak{n}} \delta_{j_p, j_q} \right).$$

Example.

$$\mathbb{E}[O_{1k_1} O_{1k_2} O_{2k_3} O_{2k_4}] = \frac{(N+1)\delta_{k_1k_2}\delta_{k_3k_4} - \delta_{k_1k_3}\delta_{k_2k_4} + \delta_{k_1k_4}\delta_{k_2k_3}}{N(N+2)(N-1)}.$$

Example of Main Theorem, again

Let $W \sim \mathrm{W}_d(\beta, \sigma)$, $(W^{ij}) = W^{-1}$, and $\gamma = \beta - \frac{d+1}{2}$.
Suppose $\gamma > 0$ but $\gamma \neq 1, 2$.

$$
\begin{aligned}
&\mathbb{E}[W^{k_1 k_2} W^{k_3 k_4} W^{k_5 k_6}] \\
&= -8 \sum_{\mathfrak{m} \in \mathcal{M}(6)} \mathrm{Wg}(\mathfrak{m}; -2\gamma) \sigma^{k_{\mathfrak{m}(1)} k_{\mathfrak{m}(2)}} \sigma^{k_{\mathfrak{m}(3)} k_{\mathfrak{m}(4)}} \sigma^{k_{\mathfrak{m}(5)} k_{\mathfrak{m}(6)}}
\end{aligned}
$$

$$
\begin{aligned}
=&\Big[\gamma(\gamma-1)(\gamma-2)(\gamma+1)(2\gamma+1)\Big]^{-1} \Big[(2\gamma^2 - 3\gamma - 1)\sigma^{k_1 k_2}\sigma^{k_3 k_4}\sigma^{k_5 k_6} \\
&+ (\gamma - 1)(\sigma^{k_1 k_3}\sigma^{k_2 k_4}\sigma^{k_5 k_6} + \sigma^{k_1 k_4}\sigma^{k_2 k_3}\sigma^{k_5 k_6} + \sigma^{k_1 k_5}\sigma^{k_2 k_6}\sigma^{k_3 k_4} \\
&+ \sigma^{k_1 k_6}\sigma^{k_2 k_5}\sigma^{k_3 k_4} + \sigma^{k_1 k_2}\sigma^{k_3 k_5}\sigma^{k_4 k_6} + \sigma^{k_1 k_2}\sigma^{k_3 k_6}\sigma^{k_4 k_5}) \\
&+ (\sigma^{k_1 k_4}\sigma^{k_2 k_5}\sigma^{k_3 k_6} + \sigma^{k_1 k_3}\sigma^{k_2 k_5}\sigma^{k_4 k_6} + \sigma^{k_1 k_4}\sigma^{k_2 k_6}\sigma^{k_3 k_5} + \sigma^{k_1 k_3}\sigma^{k_2 k_6}\sigma^{k_4 k_5} \\
&+ \sigma^{k_1 k_6}\sigma^{k_2 k_3}\sigma^{k_4 k_5} + \sigma^{k_1 k_5}\sigma^{k_2 k_3}\sigma^{k_4 k_6} + \sigma^{k_1 k_6}\sigma^{k_2 k_4}\sigma^{k_3 k_5} + \sigma^{k_1 k_5}\sigma^{k_2 k_4}\sigma^{k_3 k_6})\Big].
\end{aligned}
$$

Asymptotic expansion for $\mathrm{Wg}(g; z)$

Consider the limit $\lim_{z \to \infty} \mathrm{Wg}(g; z)$.

- Haar-distributed orthogonal matrix $O \in \mathrm{O}(N)$.

$$\mathrm{Wg}(\mathrm{id}_{S_{2n}}; N) = \mathbb{E}_{\mathrm{O}(N)}[O_{11}^2 O_{22}^2 \cdots O_{nn}^2], \qquad N \to \infty.$$

- Real Wishart matrix $W \sim \mathrm{W}_d(\beta, \sigma)$. Suppose $p = 2\beta$ is an integer.

$$W = X_1 X_1^t + X_2 X_2^t + \cdots + X_p X_p^t, \qquad p \to \infty.$$

Here the X_i are i.i.d. d-dimensional Gaussian vectors.

The leading term

Theorem. [Collins, Śniady (2006)]

Let $g \in S_{2n}$ and $\rho \vdash n$ its coset-type. Then, as $z \to \infty$, we have

$$\mathrm{Wg}(g; z) = (-1)^n \sum_{m=0}^{\infty} G_\rho(n - \ell(\rho) + m)(-z)^{-2n+\ell(\rho)-m}$$

with

$$G_\rho(n - \ell(\rho)) = \prod_{i=1}^{\ell(\rho)} \mathrm{Cat}_{\rho_i - 1}, \qquad \mathrm{Cat}_k = \frac{(2k)!}{(k+1)!k!}.$$

A combinatorial proof of this theorem was given in [M, 2010].

Combinatorial aspect

Proposition.

Let $\rho \vdash n$. Let $\mathfrak{l}_\rho$ be a fixed perfect matching of coset-type ρ. Let $G_\rho(k)$ be the coefficients in the previous theorem. Then $G_\rho(k)$ is equal to the number of sequences $(s_1, t_1, s_2, t_2, \ldots, s_k, t_k)$ of positive integers, satisfying the following three conditions.

- $t_1, \ldots, t_k$ are odd numbers and $3 \le t_1 \le t_2 \le \cdots \le t_k < 2n$.
- $s_i < t_i$ for all i.
- For the permutation $h \in S_{2n}$ given by

$$h = (s_1 \ t_1)(s_2 \ t_2) \cdots (s_k \ t_k),$$

  the perfect matching $\{\{h(1), h(2)\}, \{h(3), h(4)\}, \ldots, \{h(2n-1), h(2n)\}\}$ coincides with $\mathfrak{l}_\rho$.

The subleading term

$$(-1)^n \mathrm{Wg}(g; -z) = \prod_{i=1}^{\ell(\rho)} \mathrm{Cat}_{\rho_i - 1} \cdot z^{-2n+\ell(\rho)} + G_\rho(n-\ell(\rho)+1) z^{-2n-\ell(\rho)-1} + \cdots .$$

Theorem.

If ρ is a hook,

$$\rho = (k+1, 1^{n-k-1}),$$

then the subleading coefficient $G_\rho(n-\ell(\rho)+1)$ is given by

$$G_\rho(n-\ell(\rho)+1) = G_{(k+1,1^{n-k-1})}(k+1) = 4^k - \binom{2k+1}{k}.$$

This statement was conjectured in [M, 2010].

Very recently, Féray proved it and generalized to all partitions.

Thank you!