Unitary matrix integrals and enumerations of permutations

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Abstract.
We study an N^{-1}-expansion of a matrix integral over the unitary group $U(N)$.

Introduction

Let $U(N)$ be the unitary group of degree N:

$$U(N) = \{U = (u_{ij})_{1 \le i,j \le N} \in GL(N, \mathbb{C}) \mid UU^* = I\}.$$

There exists the unique probability measure $\mathrm{d}U$ on $U(N)$ satisfying

$$\mathrm{d}(V_1 U V_2) = \mathrm{d}U \qquad \text{for fixed } V_1, V_2 \in U(N);$$

$$\int_{U(N)} \mathrm{d}U = 1.$$

We call $\mathrm{d}U$ the normalized Haar measure on $U(N)$.
We would like to evaluate the integral of the form $\int_{U(N)} f(U)\mathrm{d}U$, where f is a polynomial function in matrix entries u_{ij} and their complex conjugates $\overline{u_{ij}}$.
Let S_n be the symmetric group acting on $\{1, 2, \ldots, n\}$.

Number Theory

We show an importance of unitary matrix integrals.
Keating and Snaith (2000) computed a unitary matrix integral:

$$\int_{U(N)} |\det(I-U)|^{2m} \mathrm{d}U = \prod_{j=1}^{N-1} \frac{j!(j+2m)!}{((j+m)!)^2} \quad \sim \left[\prod_{j=0}^{m-1} \frac{j!}{(m+j)!}\right] N^{m^2}$$

as $N \to \infty$.
This is closely related to the moment of the Riemann zeta.
Keating-Snaith conjecture:

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2m} \mathrm{d}t \overset{?}{\sim} \left[\prod_{j=0}^{m-1} \frac{j!}{(m+j)!}\right] a(m)(\log T)^{m^2},$$

as $T \to \infty$, where $a(m)$ is a number-theoretic part.

General moments for a random unitary matrix

The Weingarten calculus developed since [Collins (2003)] gives a useful technique for the unitary matrix integral computation.

Given indices i_k, j_k, i'_k, j'_k $(k = 1, 2, \ldots, n)$, we have

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_n j'_n}} \mathrm{d}U$$

$$= \sum_{\sigma, \tau \in S_n} \left(\prod_{k=1}^{n} \delta_{i_k, i'_{\sigma(k)}} \right) \left(\prod_{k=1}^{n} \delta_{j_k, j'_{\tau(k)}} \right) \mathrm{Wg}_n^{U(N)}(\sigma^{-1}\tau).$$

Here the unitary Weingarten function $\mathrm{Wg}_n^{U(N)}(\sigma)$ is defined by

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N + j - i)} \chi^\lambda(\sigma) \qquad (\sigma \in S_n).$$

Representation theory

A weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l), \qquad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$$

is said to be a partition of n if $|\lambda| := \sum_{i=1}^{l} \lambda_i = n$. Then we write $\lambda \vdash n$. We call $\ell(\lambda) := l$ the length of λ.

Irreducible representations of S_n are parametrized by partitions of n.
χ^λ: the irreducible character corresponding to λ.
f^λ: the dimension of the irrep corresponding to λ.

Example. Let $n = 2$ and $\sigma = (1\ 2)$. Then

$$\mathrm{Wg}_2^{U(N)}((1\ 2)) = \frac{1}{2!}\left(\frac{1}{N(N+1)} \cdot 1 + \frac{1}{N(N-1)} \cdot (-1)\right) = \frac{-1}{N(N^2-1)}.$$

Main Result (1)

We evaluate coefficients in the asymptotic expansion of $\mathrm{Wg}_n^{U(N)}(\sigma)$ in the limit $N \to \infty$.

Theorem [M-Novak (2009 preprint)]

Suppose $N \geq n$. For each $\sigma \in S_n$ of cycle type μ, we have

$$\mathrm{Wg}_n^{U(N)}(\sigma) = (-1)^{n-\ell(\mu)} \sum_{k=0}^{\infty} \frac{a_{n-\ell(\mu)+2k}(\sigma)}{N^{2n-\ell(\mu)+2k}}$$

with

$$a_{n-\ell(\mu)}(\sigma) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}(\mu_i - 1), \qquad \mathrm{Cat}(m) = \frac{(2m)!}{(m+1)!m!}$$

and non-negative integers $a_{n-\ell(\mu)+2k}(\sigma)$. Furthermore...

Main Result (2)

Furthermore, the non-negative integer $a_p(\sigma)$ ($p \geq 0,\ \sigma \in S_n$) coincides with the number of sequences $(s_1, t_1, s_2, t_2, \dots, s_p, t_p) \in \{1, 2, \dots, n\}^{\times 2p}$ satisfying

- $s_i < t_i$ for all i.
- $1 < t_1 \leq t_2 \leq \cdots \leq t_p \leq n$.
- $\sigma = (s_1\ t_1)(s_2\ t_2)\cdots(s_p\ t_p)$.

Example. The value $a_2(\mathrm{id}_n)$ coincides with the number of seqeunces (s_1, t_1, s_2, t_2) satisfying

$$s_1 < t_1, \quad s_2 < t_2, \quad 2 \leq t_1 \leq t_2 \leq n, \qquad \mathrm{id}_n = (s_1\ t_1)(s_2\ t_2).$$

We have $s_1 = s_2$ and $t_1 = t_2$. Hence $a_2(\mathrm{id}_n)$ equals the number of transpositions $(s_1\ t_1)$ in S_n, i.e., $a_2(\mathrm{id}_n) = \binom{n}{2}$.

Main Example

We obtain the expressions

$$\mathrm{Wg}_n^{U(N)}(\mathrm{id}_n) = \int_{U(N)} |u_{11}u_{22}\cdots u_{nn}|^2 \mathrm{d}U$$

$$= \frac{1}{n!} \sum_{\lambda \vdash n} \frac{(f^\lambda)^2}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N+j-i)}$$

$$= \sum_{k=0}^{\infty} \frac{a_{2k}(\mathrm{id}_n)}{N^{n+2k}}$$

$$= 1N^{-n} + \binom{n}{2} N^{-n-2} + \left[3\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2}\right] N^{-n-4} + O(N^{-n-6}).$$

Remark. Our main result is closely related to Jucys-Murphy elements $J_k = (1\ k) + (2\ k) + \cdots + (k-1\ k)$, which belong to the group algebra of S_n.

Closing Remarks

The Weingarten calculus on the orthogonal group $O(N)$ has been deveoloped in [Collins-Śniady (2006)] and [Collins-M (2009)]. The connection to an enumeration of permutations was given in [M (2010 preprint)]. However we need pairings (perfect matchings) instead of permutations.

Reference

1. S. Matsumoto and J. Novak, Jucys-Murphy elements and unitary matrix integrals, arXiv:0905.1992v2, 44pp.
2. S. Matsumoto, Jucys-Murphy elements, orthogonal matrix integrals, and Jack measures, arXiv:1001.2345v1, 35 pp.

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