Unitary matrix integrals and enumerations of permutations

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1. Unitary matrix integrals
2. Weingarten calculus on $U(N)$
3. Asymptotic behavior for Weingarten functions (Main Result)
4. Orthogonal matrix case

Unitary matrix

The unitary group:

$$U(N) = \{U = (u_{ij})_{1 \le i,j \le N} \in GL(N, \mathbb{C}) \mid UU^* = I\}.$$

Normalized Haar measure $\mathrm{d}U$:
there exists the unique probability measure $\mathrm{d}U$ on $U(N)$ satisfying

$$\mathrm{d}(V_1 U V_2) = \mathrm{d}U \qquad \text{for fixed } V_1, V_2 \in U(N);$$

$$\int_{U(N)} 1 \mathrm{d}U = 1.$$

General problem

Given an integrable function f on $U(N)$, we would like to compute tha value of

$$\int_{U(N)} f(U)\mathrm{d}U.$$

Furthermore, given a sequence $\{f_N\}$ of integrable functions f_N on $U(N)$, we would like to evaluate the asymptotics of

$$\int_{U(N)} f_N(U)\mathrm{d}U = a_0 N^p + a_1 N^{p-1} + a_2 N^{p-2} + \cdots \qquad (N \to \infty)$$

with some constants $a_0, a_1, a_2, \cdots \in \mathbb{C}$ and $p \in \mathbb{Z}$.

Example. Consider $f_N(U) := |u_{11} u_{22}|^2$ for $U = (u_{ij})_{1 \le i,j \le N}$. Then we will have

$$\int_{U(N)} |u_{11} u_{22}|^2 \mathrm{d}U = \frac{1}{N^2 - 1} = \frac{1}{N^2} + \frac{1}{N^4} + \frac{1}{N^6} + \cdots \sim N^{-2}.$$

Class functions

Definition.

A function f on $U(N)$ is said to be a class function if it satisfies

$$f(V^{-1}UV) = f(V^*UV) = f(U) \qquad \text{(for all } U, V \in U(N)).$$

Any unitary matrix can be diagonalized by a unitary matrix. Hence f is a class function if and only if $f(U)$ depends on eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}$ of U, i.e., f satisfies

$$f(U) = f(\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})).$$

Weyl's integration formula

If f is an integrable class function, we can compute $\int_{U(N)} f(U)\mathrm{d}U$ by the following formula.

Theorem (Weyl)

For any integrable class function f on $U(N)$, we have

$$\int_{U(N)} f(U)\mathrm{d}U = \frac{1}{(2\pi)^N N!} \int_{[0,2\pi)^N} f(\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})) \times \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_N.$$

Moments of the characteristic polynomial

Consider the integrable class function $f(U) = |\det(I - U)|^{2m}$ with $m \in \mathbb{N}$.

Theorem [Keating-Snaith (2000)]

$$\int_{U(N)} |\det(I - U)|^{2m} \mathrm{d}U = \prod_{j=1}^{N-1} \frac{j!(j+2m)!}{((j+m)!)^2}.$$

Hence we obtain its asymptotic behavior

$$\int_{U(N)} |\det(I - U)|^{2m} \mathrm{d}U \sim \left[\prod_{j=0}^{m-1} \frac{j!}{(m+j)!} \right] N^{m^2} \qquad (N \to \infty).$$

Note: This quantity is related to moments of the Riemann zeta-function: $\int_0^T |\zeta(\frac{1}{2} + t\sqrt{-1})|^{2m} \mathrm{d}t$ as $T \to \infty$.

A problem

How can we compute the integral $\int_{U(N)} f(U)\mathrm{d}U$ unless f is a class function ?

In general, we would like to compute an integral for a polynomial function of the form

$$f(U) = \left(\sum_{i,j=1}^{N} a_{ij} u_{ij}^{m_{ij}}\right)\left(\sum_{i,j=1}^{N} b_{ij} \overline{u_{ij}}^{n_{ij}}\right)$$

with $a_{ij}, b_{ij} \in \mathbb{C}$ and $m_{ij}, n_{ij} \in \mathbb{N}$. To do it, we will observe integrals

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_m j'_m}} \mathrm{d}U$$

where i_k, j_k, i'_k, j'_k are indices from $\{1, 2, \dots, N\}$.

Weingarten calculus

The calculus for the integral

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_m j'_m}} \mathrm{d}U$$

was first studied by D. Weingarten (1978), and developed by B. Collins with his coauthors since 2003.
The calculus is called the Weingarten calculus.

Claim.

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_m j'_m}} \mathrm{d}U = 0$$

unless $n = m$.

Let S_n be the symmetric group acting on $\{1, 2, \dots, n\}$.

Theorem [Collins (2003)]

$$\int_{U(N)} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_n j_n} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_n j'_n}} \mathrm{d}U$$

$$= \sum_{\sigma, \tau \in S_n} \left(\prod_{k=1}^{n} \delta_{i_k, i'_{\sigma(k)}} \right) \left(\prod_{k=1}^{n} \delta_{j_k, j'_{\tau(k)}} \right) \mathrm{Wg}_n^{U(N)}(\sigma^{-1}\tau).$$

Here the function $\mathrm{Wg}_n^{U(N)}(\sigma)$ is given by

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N + j - i)} \chi^\lambda(\sigma) \qquad (\sigma \in S_n),$$

which is called the unitary Weingarten function.

Representation theory

A weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l), \qquad (\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_l > 0)$$

is said to be a partition of n if $|\lambda| := \sum_{i=1}^{l} \lambda_i = n$. Then we write $\lambda \vdash n$. We call $\ell(\lambda) := l$ the length of λ.

Irreducible representations of S_n are parametrized by partitions of n.
χ^λ: the irreducible character corresponding to λ.
f^λ: the dimension of the irrep corresponding to λ.

Example. Let $n = 2$ and $\sigma = (1\ 2)$. Then

$$\mathrm{Wg}_2^{U(N)}((1\ 2)) = \frac{1}{2!}\left(\frac{1}{N(N+1)} \cdot 1 + \frac{1}{N(N-1)} \cdot (-1)\right) = \frac{-1}{N(N^2-1)}.$$

Example

$$\int_{U(N)} u_{11} u_{12}^2 \overline{u_{12} u_{22}} \mathrm{d}U = 0.$$

$$\int_{U(N)} u_{11} u_{12} u_{12} \overline{u_{11} u_{12} u_{22}} \mathrm{d}U = 0.$$

(The row indices are $(1,1,1)$ and $(1,1,2)$).

$$\int_{U(N)} u_{11} u_{12} u_{12} \overline{u_{11} u_{12} u_{12}} \mathrm{d}U = \sum_{\sigma \in S_3} \sum_{\tau \in \{\mathrm{id}_3, (2\ 3)\}} \mathrm{Wg}_3^{U(N)}(\sigma^{-1}\tau)$$

$$= 2 \sum_{\sigma \in S_3} \mathrm{Wg}_3^{U(N)}(\sigma) = \frac{2}{N(N+1)(N+2)}.$$

row indices: $(1,1,1)$ and $(1,1,1)$.
column indices: $(1,2,2)$ and $(1,2,2)$.

Example

In general, if $N \geq n$,

$$\mathrm{Wg}_n^{U(N)}(\sigma) = \int_{U(N)} \prod_{i=1}^{n} u_{ii} \overline{u_{i\sigma(i)}} \mathrm{d}U \qquad (\sigma \in S_n).$$

Example.

$$\int_{U(N)} u_{11} u_{22} u_{33} \overline{u_{12} u_{23} u_{31}} \mathrm{d}U$$

$$= \mathrm{Wg}_3^{U(N)}((1\ 2\ 3)) = \frac{2}{(N+2)(N+1)N(N-1)(N-2)}.$$

Cycle-types

Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition of n.
A permutation $\sigma \in S_n$ is said to be of cycle-type μ if the cycle decomposition of σ is of the form

$$\sigma = (i_1 \ i_2 \ \cdots i_{\mu_1})(j_1 \ j_2 \ \cdots j_{\mu_2}) \cdots .$$

Example. In S_6,

σ	μ
$(1\ 3\ 2)(4\ 6)(5)$	$(3, 2, 1)$
$(2\ 3)(5\ 6)(1)(4)$	$(2, 2, 1, 1)$
$(1\ 3\ 5\ 6\ 2\ 4)$	(6)
$\mathrm{id}_6 = (1)(2)(3)(4)(5)(6)$	$(1, 1, 1, 1, 1, 1)$

Asymptotic behavior

Theorem [M-Novak (2009 preprint)]

Suppose $N \geq n$. For each $\sigma \in S_n$ of cycle type μ, we have

$$\mathrm{Wg}_n^{U(N)}(\sigma) = (-1)^{n-\ell(\mu)} \sum_{k=0}^{\infty} \frac{a_{n-\ell(\mu)+2k}(\sigma)}{N^{2n-\ell(\mu)+2k}}$$

with

$$a_{n-\ell(\mu)}(\sigma) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}(\mu_i - 1), \qquad \mathrm{Cat}(m) = \frac{(2m)!}{(m+1)!m!}$$

and non-negative integers $a_{n-\ell(\mu)+2k}(\sigma)$. Furthermore...

Asymptotic behavior

Furthermore, the non-negative integer $a_p(\sigma)$ ($p \geq 0, \ \sigma \in S_n$) coincides with the number of sequences $(s_1, t_1, s_2, t_2, \dots, s_p, t_p) \in \{1, 2, \dots, n\}^{\times 2p}$ satisfying

- $s_i < t_i$ for all i.
- $1 < t_1 \leq t_2 \leq \cdots \leq t_p \leq n$.
- $\sigma = (s_1 \ t_1)(s_2 \ t_2) \cdots (s_p \ t_p)$.

Example. The value $a_2(\mathrm{id}_n)$ coincides with the number of seqeunces (s_1, t_1, s_2, t_2) satisfying

$$s_1 < t_1, \quad s_2 < t_2, \quad 2 \leq t_1 \leq t_2 \leq n, \qquad \mathrm{id}_n = (s_1 \ t_1)(s_2 \ t_2).$$

Then we have $s_1 = s_2$ and $t_1 = t_2$. Hence $a_2(\mathrm{id}_n)$ equals the number of transpositions $(s_1 \ t_1)$ in S_n, i.e., $a_2(\mathrm{id}_n) = \binom{n}{2}$.

Example

Consider the case of $\sigma = \mathrm{id}_n$.

$$\mathrm{Wg}_n^{U(N)}(\mathrm{id}_n) = \int_{U(N)} |u_{11}u_{22}\cdots u_{nn}|^2 \mathrm{d}U$$

$$= \frac{1}{n!} \sum_{\lambda \vdash n} \frac{(f^\lambda)^2}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (N+j-i)}$$

$$= \sum_{k=0}^{\infty} \frac{a_{2k}(\mathrm{id}_n)}{N^{n+2k}}$$

$$= 1N^{-n} + \binom{n}{2} N^{-n-2} + \left[3\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2}\right] N^{-n-4} + O(N^{-n-6}).$$

Orthogonal matrix integrals

Consider the orthogonal group

$$O(N) = \{V = (v_{ij})_{1 \le i,j \le N} \in GL(N, \mathbb{R}) \mid {}^tVV = I\}.$$

The group also has the normalized Haar measure $\mathrm{d}V$.
Consider the integral

$$\int_{O(N)} f(V)\mathrm{d}V$$

where f is a polynomial function in variables v_{ij}. The discussion is parallel to the unitary case but a bit more complicated.

The Weingarten calculus on $O(N)$ has been deveoloped in [Collins-Śniady (2006)] and [Collins-M (2009)].
The connection to an enumeration of permutations was given in [M (2010 preprint)]. However we need pairings (perfect matchings) instead of permutations.