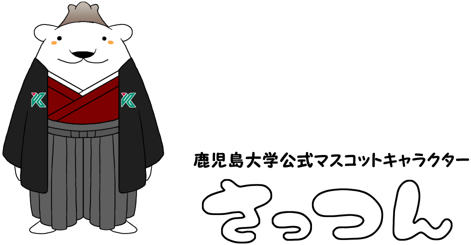
円 β アンサンブルの中心極限定理と Jack 多項式

松本 詔 (MATSUMOTO, Sho)

鹿児島大学 大学院理工学研究科
Kagoshima University



19 February, 2015

Contents

Joint Work with Tiefeng Jiang (Univ of Minnesota)
To appear in The Annals of Probability

Circular unitary ensemble (CUE)

Let $\mathbf{U}(n) = \{g \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid gg^* = I_n\}$ be the unitary group of degree n. There exists a unique probability measure μ on $\mathbf{U}(n)$ satisfying

$$\int_{\mathbf{U}(n)} f(g_1 g g_2)\mu(dg) = \int_{\mathbf{U}(n)} f(g)\mu(dg)$$

for any integrable function f on $\mathbf{U}(n)$ and fixed matrices $g_1, g_2 \in \mathbf{U}(n)$. We call μ the Haar probability measure on $\mathbf{U}(n)$.

The probability space $(\mathbf{U}(n), \mathbf{Borel}, \mu)$ is, by definition, the circular unitary ensemble (CUE).

Let U_n be an $n \times n$ random unitary matrix distributed in the Haar probability measure μ. We call U_n a CUE matrix.

Trivial Example. $(n = 1)$. $\mathbf{U}(1) = \mathbb{T} = \{g = e^{i\theta} \mid 0 \le \theta < 2\pi\}$. $\mu(dg) = \frac{d\theta}{2\pi}$.

Weyl's integral formula

Let $e^{i\theta_1}, \dots, e^{i\theta_n}$ $(\theta_1, \dots, \theta_n \in [0, 2\pi))$ be eigenvalues of U_n.
Weyl's integral formula for the unitary group claims that the density function for $e^{i\theta_1}, \dots, e^{i\theta_n}$ is given by

$$\frac{1}{(2\pi)^n n!} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

Example.

$$\mathbb{E}[|\operatorname{Tr}(U_n^2)|^4] = \int_{\mathbf{U}(n)} |\operatorname{Tr}(g^2)|^4 \mu(dg)$$

$$= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \left| \sum_{j=1}^n e^{2i\theta_j} \right|^4 \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 d\theta_2 \cdots d\theta_n.$$

Partitions

Partition of integer

- A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of $m \in \mathbb{N}$ is a weakly-decreasing sequence of positive integers with $|\lambda| = \sum_{j=1}^{l} \lambda_j = m$.
- We denote by $\ell(\lambda)$ the length of λ.
- For each positive integer r, we denote by $m_r(\lambda)$ the multiplicity of r in λ:
$$m_r(\lambda) = |\{j \mid 1 \le j \le l,\ \lambda_j = r\}|.$$
Note $m = |\lambda| = \sum_{r \ge 1} r m_r(\lambda)$ and $\ell(\lambda) = \sum_{r \ge 1} m_r(\lambda)$.
- We often write $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots)$.
- Put $z_\lambda = (\lambda_1 \cdot \lambda_2 \cdots \lambda_l) \cdot (m_1(\lambda)! \cdot m_2(\lambda)! \cdots)$.

Example. If $\lambda = (4, 2, 2, 1, 1, 1)$, then $|\lambda| = 11$, $\ell(\lambda) = 6$, and $\lambda = (1^3, 2^2, 4^1)$. Moreover, $z_\lambda = (4 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1) \cdot (3! \cdot 2! \cdot 0! \cdot 1!) = 192$.

Partitions and traces

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition. For a (random) matrix U, put

$$p_\lambda(U) = \prod_{j=1}^{l} p_{\lambda_j}(U), \qquad p_r(U) = \mathrm{Tr}\, U^r.$$

Example. $p_{(4,2,2,1,1,1)}(U) = (\mathrm{Tr}\, U^4)(\mathrm{Tr}\, U^2)^2(\mathrm{Tr}\, U)^3$.

Question.

Let U_n be an $n \times n$ CUE matrix. Let μ and ν be partitions. We shall consider the mixed moment

$$\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right].$$

This is a mixed moment for a family of random variables $\mathrm{Tr}(U_n^j), \mathrm{Tr}(U_n^{-j})$ $(j = 1, 2, \ldots)$.

Diaconis–Shahshahani Theorem

Theorem. [Diaconis–Shahshahani (1994)], [Diaconis–Evans (2001)]

For $n \geq |\mu| \vee |\nu|$,

$$\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = \delta_{\mu\nu} z_\mu.$$

This is independent of n.

Example. If $\mu = \nu = (4, 2, 2, 1, 1, 1)$ and $n \geq 11$ then

$$\mathbb{E}[|p_\mu(U_n)|^2] = z_\mu = 192.$$

If $\mu \neq \nu$ and $n \geq |\mu| \vee |\nu|$ then $\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = 0$.

CLT for CUE

$\xi^{\mathbb{C}} = \frac{1}{\sqrt{2}}(\xi^{\mathbb{R}} + i\eta^{\mathbb{R}})$, where $\xi^{\mathbb{R}}, \eta^{\mathbb{R}}$ are independent standard real normal.

Complex normal random variables

Let $\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}, \dots, \xi_k^{\mathbb{C}}$ be independent standard complex normal random variables. Let $\mu = (1^{a_1}, 2^{a_2}, \dots, k^{a_k})$ and $\nu = (1^{b_1}, 2^{b_2}, \dots, k^{b_k})$ be partitions. Then

$$\mathbb{E}\left[\prod_{j=1}^{k} (\sqrt{j}\xi_j^{\mathbb{C}})^{a_j} \overline{(\sqrt{j}\xi_j^{\mathbb{C}})^{b_j}}\right] = \delta_{\mu\nu} z_\mu.$$

Corollary. [Diaconis–Shahshahani (1994)], [Diaconis–Evans (2001)]

Let U_n be an $n \times n$ CUE matrix. For each $k \geq 1$,

$$(\mathrm{Tr}(U_n), \mathrm{Tr}(U_n^2), \dots, \mathrm{Tr}(U_n^k)) \xrightarrow{\text{dist.}} (\sqrt{1}\xi_1^{\mathbb{C}}, \sqrt{2}\xi_2^{\mathbb{C}}, \dots, \sqrt{k}\xi_k^{\mathbb{C}})$$

as $n \to \infty$.

Proof of Diaconis–Shahshahani

We use the representation theory and symmetric functions. Recall the Schur polynomial

$$s_\lambda(x_1, \dots, x_n) = \frac{\det(x_j^{\lambda_i + n - i})}{\det(x_j^{n-i})} \qquad (\ell(\lambda) \le n).$$

For an $n \times n$ matrix A with eigenvalues $a_1, \dots, a_n$, we put

$$s_\lambda(A) = s_\lambda(a_1, \dots, a_n)$$

if $\ell(\lambda) \le n$, and $s_\lambda(A) = 0$ otherwise.

First, we use the Frobenius formula

$$p_\mu = \sum_\lambda \chi_\mu^\lambda s_\lambda,$$

where $\chi_\mu^\lambda \in \mathbb{Z}$ are character values for symmetric groups. Then

$$\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = \sum_{\lambda,\rho} \chi_\mu^\lambda \chi_\nu^\rho \mathbb{E}\left[s_\lambda(U_n)\overline{s_\rho(U_n)}\right]$$

Proof of Diaconis–Shahshahani

Second, we use the orthogonal relation for Schur polynomials

$$\mathbb{E}[s_\lambda(U_n)\overline{s_\rho(U_n)}] = \delta(\ell(\lambda) \le n) \cdot \delta_{\lambda\rho}.$$

Then, since $|\lambda| = |\rho| = |\mu| = |\nu| \le n$,

$$\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = \sum_\lambda \chi_\mu^\lambda \chi_\nu^\lambda.$$

Third, we use the character relation of the second kind for symmetric groups. Then

$$\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = z_\mu \delta_{\mu\nu}.$$

This completes the proof of Diaconis–Shahshahani Theorem.

Circular orthogonal/symplectic ensembles

Remark. Diaconis–Shahshahani shows a similar formula for the orthogonal group and symplectic group.

Let $V_N^{(2)} = U_N$ be an $N \times N$ CUE matrix.
$V_n^{(1)} = U_n {U_n}^{\mathrm{T}}$: a COE matrix ($\beta = 1$).
$V_n^{(4)} = U_{2n} U_{2n}^{\mathrm{D}}$: a CSE matrix ($\beta = 4$).
Here ${U_{2n}}^{\mathrm{D}} = J_n {U_{2n}}^{\mathrm{T}} {J_n}^{\mathrm{T}}$ and $J_n = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$.

The density function for eigenvalues of $V_n^{(\beta)}$ ($\beta = 1, 2, 4$) is given by

$$\frac{\Gamma(1+\beta/2)^n}{(2\pi)^n \Gamma(1+\beta n/2)} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^{\beta}.$$

Probability spaces for three classes of random matrices $V_n^{(\beta)}$ ($\beta = 1, 2, 4$) are called Dyson's circular ensembles (1962).

Circular β-ensembles $\mathrm{C}\beta\mathrm{E}_n$ with general $\beta > 0$

Space: $\{(e^{i\theta_1}, \dots, e^{i\theta_n})\} = \mathbb{T}^{\times n} \cong [0, 2\pi)^{\times n}$.

Density: $$\frac{\Gamma(1+\beta/2)^n}{(2\pi)^n \Gamma(1+\beta n/2)} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

- This is the eigenvalue density for circular orthogonal/unitary/symplectic ensembles for $\beta = 1/2/4$, respectively.
- Matrix expression $V_n^{(\beta)}$ for general $\beta > 0$: [Killip–Nenciu (2004)]

$Z_n^{(\beta)} = (e^{i\theta_1}, \dots, e^{i\theta_n})$: A random array taken from the CβE.

We are interested in "traces of powers"

$$p_r(Z_n^{(\beta)}) = \sum_{j=1}^n e^{ir\theta_j} = \mathrm{Tr}\left[(V_n^{(\beta)})^r\right] \qquad (r = 1, 2, 3, \dots).$$

CβE case – prologue –

Question.

Extend the Diaconis–Shahshahani theorem to the CβE with general $\beta > 0$. In other words, compute

$$\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$$
$$= \int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{r=1}^{\ell(\mu)}\left(\sum_{j=1}^{n} e^{i\mu_r\theta_j}\right)\cdot\prod_{r=1}^{\ell(\nu)}\left(\sum_{j=1}^{n} e^{i\nu_r\theta_j}\right)$$
$$\times\frac{\Gamma(1+\beta/2)^n}{(2\pi)^n\Gamma(1+\beta n/2)}\prod_{1\le j<k\le n}|e^{i\theta_j}-e^{i\theta_k}|^\beta d\theta_1\cdots d\theta_n$$

Recall the $\beta = 2$ case: If $n \ge |\mu| = |\nu|$ then
$\mathbb{E}\left[p_\mu(Z_n^{(2)})\overline{p_\nu(Z_n^{(2)})}\right] = z_\mu\delta_{\mu\nu}$.

Put $\alpha = 2/\beta$.

Example. Consider the average of $|p_2(Z_n^{(\beta)})|^2$. Note that $|p_2(Z_n^{(\beta=2)})|^2 = |\operatorname{Tr}(U_n^2)|^2$. For $n \geq 2$,

$$\mathbb{E}\left[|p_2(Z_n^{(\beta)})|^2\right] = \frac{2\alpha n(n^2 + 2(\alpha-1)n + \alpha^2 - 3\alpha + 1)}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)}$$

$$= \begin{cases} \frac{4(n^2+2n-1)}{(n+1)(n+3)} & \text{if } \beta = 1 \text{ i.e. } \alpha = 2 \text{ (COE case)} \\ 2 & \text{if } \beta = 2 \text{ i.e. } \alpha = 1 \text{ (CUE case)} \\ \frac{4n^2-4n-1}{(2n-1)(2n-3)} & \text{if } \beta = 4 \text{ i.e. } \alpha = 1/2 \text{ (CSE case).} \end{cases}$$

CβE case – prologue –

Example. For $n \geq 2$,

$$\mathbb{E}\left[p_2(Z_n^{(\beta)})\overline{p_{(1,1)}(Z_n^{(\beta)})}\right] = \frac{2\alpha^2(\alpha-1)n}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)}$$

$$= \begin{cases} \frac{8}{(n+1)(n+3)} & \text{if } \beta = 1 \text{ i.e. } \alpha = 2 \text{ (COE case)} \\ 0 & \text{if } \beta = 2 \text{ i.e. } \alpha = 1 \text{ (CUE case)} \\ \frac{-1}{(2n-1)(2n-3)} & \text{if } \beta = 4 \text{ i.e. } \alpha = 1/2 \text{ (CSE case).} \end{cases}$$

Thus, except $\beta = 2$, even if $n \geq |\mu| \vee |\nu|$ and $\mu \neq \nu$, the moments $\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$ can be nonzero. In general, $\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$ would be quite complicated.

Strategy

We give up on the fulfillment of the explicit expression for $\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$.

Main Result 1

Let $\beta > 0$ and set $\alpha = 2/\beta$. Let m be a positive integer. Define two constants

$$A = A(n,m,\alpha) = \left(1 - \frac{|\alpha - 1|}{n - m + \alpha}\delta(\alpha \geq 1)\right)^m,$$

$$B = B(n,m,\alpha) = \left(1 + \frac{|\alpha - 1|}{n - m + \alpha}\delta(\alpha < 1)\right)^m.$$

Remark: $A \leq B$. Either $A = 1$ or $B = 1$ for each α.
$A(n,m,1) = B(n,m,1) = 1$.

We shall obtain an upper bound and lower bound for

$$\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$$

Main Result 1

Theorem 1. [Jiang–M (in press)]

(i) If μ is a partition of m and $n \geq m$, then

$$A \leq \frac{\mathbb{E}\left[|p_\mu(Z_n^{(\beta)})|^2\right]}{\alpha^{\ell(\mu)} z_\mu} \leq B.$$

(ii) If μ and ν are distinct partitions of m and $n \geq m$, then

$$\left|\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]\right| \leq (|A-1| \vee |B-1|) \cdot \sqrt{\alpha^{\ell(\mu)+\ell(\nu)} z_\mu z_\nu}.$$

(iii) If μ and ν are partitions with $|\mu| \neq |\nu|$, then

$$\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right] = 0.$$

Substituting $\beta = 2$ reduces this to the Diaconis–Shahshahani theorem.

Main Result 1

If we evaluate constants A and B, we obtain the following corollary.

Corollary.

(i) If μ is a partition of m and $n \geq m$, then

$$\left| \mathbb{E}\left[|p_\mu(Z_n^{(\beta)})|^2 \right] - \alpha^{\ell(\mu)} z_\mu \right| \leq 6|\alpha - 1| \frac{m}{n} \alpha^{\ell(\mu)} z_\mu.$$

(ii) If μ and ν are distinct partitions of m and $n \geq m$, then

$$\left| \mathbb{E}\left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] \right| \leq 6|\alpha - 1| \frac{m}{n} \sqrt{\alpha^{\ell(\mu)+\ell(\nu)} z_\mu z_\nu}.$$

In particular, for μ, ν fixed,

$$\lim_{n\to\infty} \mathbb{E}\left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] = \delta_{\mu\nu} \left(\frac{2}{\beta} \right)^{\ell(\mu)} z_\mu.$$

CLT for CβE

Corollary.

Let $\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}, \dots$ be independent standard complex normal random variables. For each $k \geq 1$, the random vector

$$(p_1(Z_n^{(\beta)}), p_2(Z_n^{(\beta)}), \dots, p_k(Z_n^{(\beta)}))$$

converges in distribution to

$$(\sqrt{\frac{2 \cdot 1}{\beta}}\xi_1^{\mathbb{C}}, \sqrt{\frac{2 \cdot 2}{\beta}}\xi_2^{\mathbb{C}}, \dots, \sqrt{\frac{2k}{\beta}}\xi_k^{\mathbb{C}})$$

as $n \to \infty$.

Remark that this corollary is not new.

Outline of the proof of Theorem 1: Jack polynomials

The idea of the proof is the same with that of Diaconis–Shahshahani. But we replace the Schur polynomial s_λ by the Jack polynomial $J_\lambda^{(\alpha)}$. The Jack polynomial is a deformation of the Schur polynomial with a Jack parameter α.

$$J_\lambda^{(1)} = (\text{hook-length product of } \lambda) \times s_\lambda.$$

The analytic orthogonal relation for Jack polynomials:

$$\mathbb{E}\left[J_\lambda{}^{(2/\beta)}(Z_n^{(\beta)}) \overline{J_\mu{}^{(2/\beta)}(Z_n^{(\beta)})} \right] = \delta_{\lambda\mu} \cdot \delta(\ell(\lambda) \le n) \cdot C_\lambda(2/\beta) \cdot \mathcal{N}_\lambda^{2/\beta}(n),$$

where

$$C_\lambda(\alpha) = \prod_{(i,j)\in\lambda} (\alpha(\lambda_i - j) + \lambda'_j - i + 1)(\alpha(\lambda_i - j) + \lambda'_j - i + \alpha),$$

$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}. \qquad (\mathcal{N}_\lambda^{\alpha=1}(n) \equiv 1)$$

Outline of the proof of Theorem 1: Algebraic expression

Expansion of the power-sum p_μ in Jack polynomials:

$$p_\mu = \alpha^{\ell(\mu)} z_\mu \sum_{\lambda : |\lambda| = |\mu|} \frac{\theta^\lambda_\mu(\alpha)}{C_\lambda(\alpha)} J^{(\alpha)}_\lambda,$$

where $\theta^\lambda_\mu(\alpha)$ is the Jack character.
$\theta^\lambda_\mu(1) = z_\mu^{-1} \cdot (\text{hook-length product of } \lambda) \cdot \chi^\lambda_\mu$.

Proposition (Algebraic Expression).

If μ and ν are partitions of m, then

$$\mathbb{E}\left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] = \alpha^{\ell(\mu)+\ell(\nu)} z_\mu z_\nu \sum_{\substack{\lambda : |\lambda| = m, \\ \ell(\lambda) \le n}} \frac{\theta^\lambda_\mu(\alpha)\theta^\lambda_\nu(\alpha)}{C_\lambda(\alpha)} \mathcal{N}^\alpha_\lambda(n).$$

Outline of the proof of Theorem 1: Evaluation of $\mathcal{N}_\lambda^\alpha(n)$

Recall

$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n+(j-1)\alpha-(i-1)}{n+j\alpha-i}.$$

$$A(n,m,\alpha) = \left(1-\frac{|\alpha-1|}{n-m+\alpha}\delta(\alpha\geq 1)\right)^m,$$

$$B(n,m,\alpha) = \left(1+\frac{|\alpha-1|}{n-m+\alpha}\delta(\alpha< 1)\right)^m.$$

Lemma (Elementary estimation).

Assume $n \geq m$. For any partition λ of m,

$$A(n,m,\alpha) \ \leq\ \mathcal{N}_\lambda^\alpha(n) \ \leq\ B(n,m,\alpha).$$

$$|\mathcal{N}_\lambda^\alpha(n)-1| \ \leq\ |A(n,m,\alpha)-1| \vee |B(n,m,\alpha)-1|.$$

Outline of the proof of Theorem 1: Jack characters

Show only an upper bound in the $\mu = \nu$ case. We have obtained

$$\mathbb{E}\left[|p_\mu(Z_n^{(\beta)})|^2\right] \le B(n,m,\alpha) \cdot (\alpha^{\ell(\mu)} z_\mu)^2 \sum_{\substack{\lambda : |\lambda| = m, \\ \ell(\lambda) \le n}} \frac{\theta_\mu^\lambda(\alpha)\theta_\mu^\lambda(\alpha)}{C_\lambda(\alpha)}.$$

We now suppose $n \ge m$. Using the orthogonal relation for Jack characters

$$\sum_\lambda \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} = \delta_{\mu\nu}(\alpha^{\ell(\mu)} z_\mu)^{-1},$$

we finally obtain

$$\mathbb{E}\left[|p_\mu(Z_n^{(\beta)})|^2\right] \le \alpha^{\ell(\mu)} z_\mu B(n,m,\alpha).$$

Linear combination of traces ($\beta = 2$)

Let $\xi^{\mathbb{C}}$ be a complex standard normal random variable.

Theorem. [Diaconis–Evans (2001)]

Let U_n be an $n \times n$ CUE matrix. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{j=1}^{\infty} j|a_j|^2 = \sigma^2 < \infty$. Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(U_n) = \sum_{j=1}^{\infty} a_j \operatorname{Tr}[(U_n)^j] \quad \xrightarrow{\text{dist.}} \quad \sigma \xi^{\mathbb{C}}.$$

Note: Diaconis–Evans obtained a slightly stronger version.

Key lemma (easy proof).

$$\mathbb{E}\left[|p_m(U_n)|^2\right] = m \wedge n \qquad \text{for all positive integers } m, n.$$

Proof of $\mathbb{E}\left[|p_m(U_n)|^2\right] = m \wedge n$

For any $m, n \in \mathbb{N}$, we have

$$\mathbb{E}\left[|p_m(U_n)|^2\right] = \sum_{\substack{\lambda:|\lambda|=m, \\ \ell(\lambda)\le n}} |\chi^\lambda_{(m)}|^2.$$

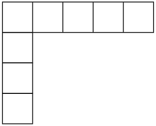
It is well known that

$$\chi^\lambda_{(m)} = \begin{cases} (-1)^{\ell(\lambda)-1} & \text{if } \lambda \text{ is a hook,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathbb{E}\left[|p_m(U_n)|^2\right] = \#\{\text{hooks with } m \text{ boxes and with height } \le n\} = m \wedge n.$$

a hook



$(\lambda_2 \le 1)$

Linear combination of traces ($\beta = 1$)

Theorem 2. [Jiang–M (in press)] CLT for COE

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying

$$\sum_{j=1}^{\infty} j|a_j|^2 = \sigma^2 < \infty.$$

Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(Z_n^{(\beta=1)}) \quad \xrightarrow{\text{dist.}} \quad \sqrt{2}\sigma\xi^{\mathbb{C}}.$$

Key lemma (quite complicated proof).

There exists a universal constant K such that

$$\mathbb{E}\left[|p_m(Z_n^{(\beta=1)})|^2\right] \leq Km \qquad \text{for all positive integers } m, n.$$

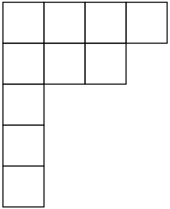
Strategy of proof of $\mathbb{E}\left[|p_m(Z_n^{(\beta=1)})|^2\right] \le Km$

Set $\beta = 1$. Then $\alpha = 2/\beta = 2$. For any $m, n \in \mathbb{N}$,

$$\mathbb{E}\left[|p_m(Z_n^{(\beta=1)})|^2\right] = (2m)^2 \sum_{\substack{\lambda:|\lambda|=m,\\ \ell(\lambda)\le n}} \frac{|\theta^{\lambda}_{(m)}(2)|^2}{C_\lambda(2)} \mathcal{N}^{(2)}_\lambda(n).$$

It is well known that

$$\theta^{\lambda}_{(m)}(2) = \begin{cases} \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (2j - i - 1) & \text{if } \lambda_3 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$


,

$\times$	2	4	6
-1	1	3	
-2			
-3			
-4			

, $\theta^{(4,3,1,1,1)}_{(10)}(2) = 3456$

Strategy of proof of $\mathbb{E}\left[|p_m(Z_n^{(\beta=1)})|^2\right] \le Km$

From our Theorem 1, we have: $\mathbb{E}\left[|p_m(Z_n^{(\beta=1)})|^2\right] \le 2m$ for $m \le n$.

Suppose $n < m$. Elementary (but involved) estimation gives

$$(2m)^2 \frac{|\theta^{(m)}_{(m)}(2)|^2}{C_{(m)}(2)} \mathcal{N}^{(2)}_{(m)}(n) < C_1\sqrt{n};$$

$$(2m)^2 \sum_{\substack{\lambda=(m-r,r)\\ 1\le r\le m/2}} \frac{|\theta^{\lambda}_{(m)}(2)|^2}{C_{\lambda}(2)} \mathcal{N}^{(2)}_{\lambda}(n) < C_2\sqrt{n};$$

$$(2m)^2 \sum_{\substack{\lambda=(r,s,1^{m-r-s})\\ r\ge s\ge 1\\ 3\le \ell(\lambda)=m-r-s+2\le n}} \frac{|\theta^{\lambda}_{(m)}(2)|^2}{C_{\lambda}(2)} \mathcal{N}^{(2)}_{\lambda}(n) < C_3 n;$$

Linear combination of traces ($\beta = 4$)

Theorem 3. [Jiang–M (in press)] CLT for CSE

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{j=1}^{\infty}(j \log j)|a_j|^2 < \infty$. Set $\sigma^2 = \sum_{j=1}^{\infty} j|a_j|^2$. Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(Z_n^{(\beta=4)}) \xrightarrow{\text{dist.}} \frac{1}{\sqrt{2}}\sigma \xi^{\mathbb{C}}.$$

Key lemma (quite complicated proof).

There exists a universal constant K such that

$$\mathbb{E}\left[|p_m(Z_n^{(\beta=4)})|^2\right] \le K m \log m \qquad \text{for all positive integers } m, n.$$

Remark: There exist universal constants K_1, K_2 such that (if $n = m$)
$K_1 m \log m \le \mathbb{E}\left[|p_m(Z_m^{(\beta=4)})|^2\right] \le K_2 m \log m.$

Future works

Conjecture

Let β be any positive real number. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{j=1}^{\infty} j|a_j|^2 =: \sigma^2 < \infty$. Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(Z_n^{(\beta)}) \quad \xrightarrow{\text{dist.}} \quad \sqrt{\frac{2}{\beta}}\sigma\xi^{\mathbb{C}}.$$

Consider a Macdonald version.

Jack polynomial $\alpha > 0$ $\xrightarrow{\textit{generalization}}$ Macdonald polynomial $q, t \in (-1, 1)$

$$(\text{const.}) \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^{\beta} \xrightarrow{\textit{generalization}}$$

$$(\text{const.}) \prod_{1\le j<k\le n} \prod_{r=0}^{\infty} \left| \frac{1 - q^r e^{i(\theta_j - \theta_k)}}{1 - tq^r e^{i(\theta_j - \theta_k)}} \right|^2$$

Future works

Generalize the following theorem to general $\beta > 0$.

Theorem [Johansson (1997)]

Let U_n be an $n \times n$ CUE matrix. Let $a_1, \dots, a_m, b_1, \dots, b_m \in \mathbb{R}$ and consider the random variable

$$X_n = \sum_{j=1}^{m} (a_j \operatorname{Re} \operatorname{Tr}[(U_n)^j] + b_j \operatorname{Im} \operatorname{Tr}[(U_n)^j]).$$

Set $\sigma^2 := \mathrm{Var}[X_n] = \frac{1}{2} \sum_{j=1}^{m} j(a_j^2 + b_j^2)$. Let $F_n(x)$ be the distribution function of the random variable $\sigma^{-1} X_n$, $\Phi(x)$ the standard normal distribution function: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Then there are positive constants C and δ such that

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \le C n^{-\delta n} \qquad \text{for all } n \ge 1.$$

Thank you for listening.