円 β アンサンブルのトレースの中心極限定理

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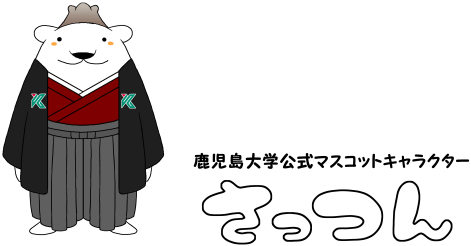

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Circular unitary ensemble (CUE)

Let $\mathbf{U}(n) = \{g \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid gg^* = I_n\}$ be the unitary group of degree n. There exists a unique probability measure μ on $\mathbf{U}(n)$ satisfying

$$\int_{\mathbf{U}(n)} f(g_1 g g_2)\mu(dg) = \int_{\mathbf{U}(n)} f(g)\mu(dg)$$

for any integrable function f on $\mathbf{U}(n)$ and fixed matrices $g_1, g_2 \in \mathbf{U}(n)$. We call μ the Haar probability measure on $\mathbf{U}(n)$.

Let U_n be an $n \times n$ random unitary matrix distributed in the Haar probability measure μ. We call U_n a CUE matrix.

Let $e^{i\theta_1}, \dots, e^{i\theta_n}$ $(\theta_1, \dots, \theta_n \in [0, 2\pi))$ be eigenvalues of U_n. Weyl's integral formula for the unitary group claims that the density function for $e^{i\theta_1}, \dots, e^{i\theta_n}$ is given by

$$\frac{1}{(2\pi)^n n!} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

Circular orthogonal/symplectic ensembles

Let $V_N^{(2)} = U_N$ be an $N \times N$ CUE matrix.
$V_n^{(1)} = U_n U_n{}^{\mathrm{T}}$: a COE matrix ($\beta = 1$).
$V_n^{(4)} = U_{2n} U_{2n}^{\mathrm{D}}$: a CSE matrix ($\beta = 4$).
Here $U_{2n}{}^{\mathrm{D}} = J_n U_{2n}{}^{\mathrm{T}} J_n{}^{\mathrm{T}}$ and $J_n = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$.

The density function for eigenvalues of $V_n^{(\beta)}$ ($\beta = 1, 2, 4$) is given by

$$\frac{\Gamma(1+\beta/2)^n}{(2\pi)^n \Gamma(1+\beta n/2)} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

Probability spaces for three classes of random matrices $V_n^{(\beta)}$ ($\beta = 1, 2, 4$) are called Dyson's circular ensembles.

Circular β-ensembles $\mathrm{C}\beta\mathrm{E}_n$ with general $\beta > 0$

Space: $\{(e^{i\theta_1}, \dots, e^{i\theta_n})\} = (S^1)^{\times n} \cong [0, 2\pi)^{\times n}$.

Density: $$\frac{\Gamma(1+\beta/2)^n}{(2\pi)^n \Gamma(1+\beta n/2)} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

- This is the eigenvalue density for circular orthogonal/unitary/symplectic ensembles for $\beta = 1/2/4$, respectively.
- Matrix expression $V_n^{(\beta)}$ for general $\beta > 0$: [Killip–Nenciu (2004)]

$Z_n^{(\beta)} = (e^{i\theta_1}, \dots, e^{i\theta_n})$: A random vector taken from the CβE.

We are interested in "traces of powers"

$$p_r(Z_n^{(\beta)}) = \sum_{j=1}^n e^{ir\theta_j} = \mathrm{Tr}\left[(V_n^{(\beta)})^r\right] \qquad (r = 1, 2, 3, \dots).$$

Partitions

Partition of integer

- A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of $m \in \mathbb{N}$ is a weakly-decreasing sequence of positive integers with $|\lambda| = \sum_{j=1}^{l} \lambda_j = m$.
- We denote by $\ell(\lambda)$ the length of λ.
- For each positive integer r, we denote by $m_r(\lambda)$ the multiplicity of r in λ:
$$m_r(\lambda) = |\{j \mid 1 \le j \le l,\ \lambda_j = r\}|.$$
Note $m = |\lambda| = \sum_{r \ge 1} r m_r(\lambda)$ and $\ell(\lambda) = \sum_{r \ge 1} m_r(\lambda)$.
- We often write $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots)$.
- Put
$$z_\lambda = \prod_{r \ge 1} r^{m_r(\lambda)} m_r(\lambda)!.$$

Example: If $\lambda = (4, 2, 2, 1, 1, 1)$, then $|\lambda| = 11$, $\ell(\lambda) = 6$, and $\lambda = (1^3, 2^2, 4^1)$. Moreover, $z_\lambda = (1^3 \cdot 3!) \times (2^2 \cdot 2!) \times (4^1 \cdot 1!) = 192$.

Products of traces

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition. For a (random) matrix U, put

$$p_\lambda(U) = \prod_{j=1}^{l} p_{\lambda_j}(U), \qquad p_r(U) = \mathrm{Tr}\, U^r.$$

For a (random) vector $Z = (Z_1, \dots, Z_n)$, put

$$p_\lambda(Z) = \prod_{j=1}^{l} p_{\lambda_j}(Z), \qquad p_r(Z) = \sum_{j=1}^{n} Z_j^r.$$

Example: If $\lambda = (4, 2, 2, 1, 1, 1) = (1^3, 2^2, 4^1)$ then

$$p_\lambda(U) = (\mathrm{Tr}\, U^4)(\mathrm{Tr}\, U^2)^2(\mathrm{Tr}\, U)^3,$$

$$p_\lambda(Z_n^{(\beta)}) = p_4(Z_n^{(\beta)}) p_2(Z_n^{(\beta)})^2 p_1(Z_n^{(\beta)})^3, \quad p_r(Z_n^{(\beta)}) = \sum_{j=1}^{n} e^{ir\theta_j}.$$

Diaconis–Shahshahani Theorem

Let U_n be an $n \times n$ CUE matrix. Let μ and ν be partitions.
The quantity $\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right]$ is a mixed moment for a family of random variables $\mathrm{Tr}(U_n^j), \mathrm{Tr}(U_n^{-j})$ $(j = 1, 2, \dots)$.

Theorem 1. [Diaconis–Shahshahani (1994)], [Diaconis–Evans (2001)]

For $n \geq |\mu| \vee |\nu|$,

$$\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = \delta_{\mu\nu} z_\mu.$$

This is independent of n.

Example: If $\mu = \nu = (4, 2, 2, 1, 1, 1)$ and $n \geq 11$ then $\mathbb{E}[|p_\mu(U_n)|^2] = 192$.
If $\mu \neq \nu$ and $n \geq |\mu| \vee |\nu|$ then $\mathbb{E}\left[p_\mu(U_n)\overline{p_\nu(U_n)}\right] = 0$.

CLT for CUE

Complex normal random variables

Let $\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}, \dots, \xi_k^{\mathbb{C}}$ be independent standard complex normal random variables. Let $\mu = (1^{a_1}, 2^{a_2}, \dots, k^{a_k})$ and $\nu = (1^{b_1}, 2^{b_2}, \dots, k^{b_k})$ be partitions. Then

$$\mathbb{E}\left[\prod_{j=1}^{k} (\sqrt{j}\xi_j^{\mathbb{C}})^{a_j} \overline{(\sqrt{j}\xi_j^{\mathbb{C}})^{b_j}}\right] = \delta_{\mu\nu} z_\mu.$$

Corollary. [Diaconis–Shahshahani (1994)], [Diaconis–Evans (2001)]

Let U_n be an $n \times n$ CUE matrix. For each $k \geq 1$,

$$(\mathrm{Tr}(U_n), \mathrm{Tr}(U_n^2), \dots, \mathrm{Tr}(U_n^k)) \xrightarrow{\text{dist.}} (\sqrt{1}\xi_1^{\mathbb{C}}, \sqrt{2}\xi_2^{\mathbb{C}}, \dots, \sqrt{k}\xi_k^{\mathbb{C}})$$

as $n \to \infty$.

$\mathrm{C}\beta\mathrm{E}$ case – prologue –

Question.

Extend the Diaconis–Shahshahani theorem to the $\mathrm{C}\beta\mathrm{E}$ with general $\beta > 0$.

Let $Z_n^{(\beta)}$ be a random vector taken in $\mathrm{C}\beta\mathrm{E}_n$. Put $\alpha = 2/\beta$.

Example: Consider the average of $|p_2(Z_n^{(\beta)})|^2$. Note that $|p_2(Z_n^{(\beta=2)})|^2 = |\operatorname{Tr}(U_n^2)|^2$. For $n \geq 2$,

$$\mathbb{E}\left[|p_2(Z_n^{(\beta)})|^2\right] = \frac{2\alpha n(n^2 + 2(\alpha-1)n + \alpha^2 - 3\alpha + 1)}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)}$$

$$= \begin{cases} \frac{4(n^2+2n-1)}{(n+1)(n+3)} & \text{if } \beta = 1 \text{ i.e. } \alpha = 2 \text{ (COE case)} \\ 2 & \text{if } \beta = 2 \text{ i.e. } \alpha = 1 \text{ (CUE case)} \\ \frac{4n^2-4n-1}{(2n-1)(2n-3)} & \text{if } \beta = 4 \text{ i.e. } \alpha = 1/2 \text{ (CSE case).} \end{cases}$$

$\mathrm{C}\beta\mathrm{E}$ case – prologue –

Example: For $n \geq 2$,

$$\mathbb{E}\left[p_2(Z_n^{(\beta)})\overline{p_{(1,1)}(Z_n^{(\beta)})}\right] = \frac{2\alpha^2(\alpha-1)n}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)}$$

$$= \begin{cases} \frac{8}{(n+1)(n+3)} & \text{if } \beta=1 \text{ i.e. } \alpha=2 \text{ (COE case)} \\ 0 & \text{if } \beta=2 \text{ i.e. } \alpha=1 \text{ (CUE case)} \\ \frac{-1}{(2n-1)(2n-3)} & \text{if } \beta=4 \text{ i.e. } \alpha=1/2 \text{ (CSE case).} \end{cases}$$

Thus, except $\beta = 2$, even if $n \geq |\mu| \vee |\nu|$ and $\mu \neq \nu$, the moments $\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$ can be nonzero. In general, $\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$ would be quite complicated.

Strategy

We give up on the fulfillment of the explicit expression for $\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right]$.

Main Result 1

Let $\beta > 0$ and set $\alpha = 2/\beta$. Let m be a positive integer. Define two constants

$$A = A(n,m,\alpha) = \left(1 - \frac{|\alpha - 1|}{n - m + \alpha}\delta(\alpha \ge 1)\right)^m,$$
$$B = B(n,m,\alpha) = \left(1 + \frac{|\alpha - 1|}{n - m + \alpha}\delta(\alpha < 1)\right)^m.$$

Remark: $A \le B$. $A(n,m,1) = B(n,m,1) = 1$.

Theorem 2. [Jiang–M (2014)]

(i) If μ is a partition of m and $n \ge m$, then

$$A \le \frac{\mathbb{E}\left[|p_\mu(Z_n^{(\beta)})|^2\right]}{\alpha^{\ell(\mu)} z_\mu} \le B.$$

Main Result 1

(ii) If μ and ν are distinct partitions of m and $n \geq m$, then

$$\left| \mathbb{E} \left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] \right| \leq (|A-1| \vee |B-1|) \cdot \sqrt{\alpha^{\ell(\mu)+\ell(\nu)} z_\mu z_\nu}.$$

(iii) If μ and ν are partitions with $|\mu| \neq |\nu|$, then

$$\mathbb{E} \left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] = 0.$$

- Substituting $\beta = 2$ reduces our theorem to the Diaconis–Shahshahani theorem $\mathbb{E} \left[p_\mu(Z_n^{(\beta=2)}) \overline{p_\nu(Z_n^{(\beta=2)})} \right] = \delta_{\mu\nu} z_\mu$.

Main Result 1

If we evaluate constants A, B, we obtain the following corollary.

Corollary.

(i) If μ is a partition of m and $n \geq m$, then

$$\left| \mathbb{E}\left[|p_\mu(Z_n^{(\beta)})|^2 \right] - \alpha^{\ell(\mu)} z_\mu \right| \leq 6|\alpha - 1| \frac{m}{n} \alpha^{\ell(\mu)} z_\mu.$$

(ii) If μ and ν are distinct partitions of m and $n \geq m$, then

$$\left| \mathbb{E}\left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] \right| \leq 6|\alpha - 1| \frac{m}{n} \sqrt{\alpha^{\ell(\mu)+\ell(\nu)} z_\mu z_\nu}.$$

In particular, for μ, ν fixed,

$$\lim_{n \to \infty} \mathbb{E}\left[p_\mu(Z_n^{(\beta)}) \overline{p_\nu(Z_n^{(\beta)})} \right] = \delta_{\mu\nu} \left(\frac{2}{\beta} \right)^{\ell(\mu)} z_\mu.$$

CLT for CβE

Corollary.

Let $\xi_1^{\mathbb{C}}, \xi_2^{\mathbb{C}}, \dots$ be independent standard complex normal random variables. For each $k \geq 1$, the random vector

$$(p_1(Z_n^{(\beta)}), p_2(Z_n^{(\beta)}), \dots, p_k(Z_n^{(\beta)}))$$

converges in distribution to

$$(\sqrt{\frac{2 \cdot 1}{\beta}} \xi_1^{\mathbb{C}}, \sqrt{\frac{2 \cdot 2}{\beta}} \xi_2^{\mathbb{C}}, \dots, \sqrt{\frac{2k}{\beta}} \xi_k^{\mathbb{C}})$$

as $n \to \infty$.

Remark that this corollary is not new.

Outline of the proof of Theorem 2

A main tool is the Jack polynomial $J_\lambda^{(\alpha)}$, which is a family of multivariate orthogonal polynomials with Jack parameter α.

$$\mathbb{E}\left[J_\mu{}^{(2/\beta)}(Z_n^{(\beta)})\overline{J_\nu{}^{(2/\beta)}(Z_n^{(\beta)})}\right] = \delta_{\mu\nu} \cdot \delta(\ell(\mu) \le n) \cdot C_\lambda(2/\beta) \cdot \mathcal{N}_\lambda^{2/\beta}(n),$$

where

$$C_\lambda(\alpha) = \prod_{(i,j)\in\lambda} (\alpha(\lambda_i - j) + \lambda'_j - i + 1)(\alpha(\lambda_i - j) + \lambda'_j - i + \alpha),$$

$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}. \qquad (\mathcal{N}_\lambda^{\alpha=1}(n) \equiv 1)$$

Note: $(\beta = 2)$

$$\mathbb{E}[s_\mu(U_n)\overline{s_\nu(U_n)}] = \delta_{\mu\nu} \cdot \delta(\ell(\mu) \le n),$$

where s_λ is the Schur polynomial.

Outline of the proof of Theorem 2

Expansion of the power-sum p_μ in Jack polynomials:

$$p_\mu = \alpha^{\ell(\mu)} z_\mu \sum_{\lambda:|\lambda|=|\mu|} \frac{\theta^\lambda_\mu(\alpha)}{C_\lambda(\alpha)} J^{(\alpha)}_\lambda,$$

where $\theta^\lambda_\mu(\alpha)$ is the Jack character.

Proposition (Algebraic Expression).

If μ and ν are partitions of m, then

$$\mathbb{E}\left[p_\mu(Z_n^{(\beta)})\overline{p_\nu(Z_n^{(\beta)})}\right] = \alpha^{\ell(\mu)+\ell(\nu)} z_\mu z_\nu \sum_{\substack{\lambda:|\lambda|=m,\\ \ell(\lambda)\le n}} \frac{\theta^\lambda_\mu(\alpha)\theta^\lambda_\nu(\alpha)}{C_\lambda(\alpha)} \mathcal{N}^\alpha_\lambda(n).$$

Evaluation of $\mathcal{N}_\lambda^\alpha(n)$

Recall

$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}.$$

$$A(n,m,\alpha) = \left(1 - \frac{|\alpha - 1|}{n - m + \alpha}\delta(\alpha \geq 1)\right)^m,$$

$$B(n,m,\alpha) = \left(1 + \frac{|\alpha - 1|}{n - m + \alpha}\delta(\alpha < 1)\right)^m.$$

Lemma (Elementary estimation).

Assume $n \geq m$. For any partition λ of m,

$$A(n,m,\alpha) \;\leq\; \mathcal{N}_\lambda^\alpha(n) \;\leq\; B(n,m,\alpha),$$
$$|\mathcal{N}_\lambda^\alpha(n) - 1| \;\leq\; |A(n,m,\alpha) - 1| \vee |B(n,m,\alpha) - 1|.$$

Linear combination of traces ($\beta = 2$)

Let $\xi^{\mathbb{C}}$ be a complex standard normal random variable.

Theorem. [Diaconis–Evans (2001)]

Let U_n be an $n \times n$ CUE matrix. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{j=1}^{\infty} j|a_j|^2 = \sigma^2 < \infty$. Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(U_n) = \sum_{j=1}^{\infty} a_j \operatorname{Tr}[(U_n)^j] \quad \xrightarrow{\text{dist.}} \quad \sigma \xi^{\mathbb{C}}.$$

Note: Diaconis–Evans obtained a slightly stronger version.

Key lemma (easy proof).

$$\mathbb{E}\left[|p_m(U_n)|^2\right] = m \wedge n \qquad \text{for all positive integers } m, n.$$

Linear combination of traces ($\beta = 1$)

Theorem 3. [Jiang–M (2014)] CLT for COE

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying

$$\sum_{j=1}^{\infty} j|a_j|^2 = \sigma^2 < \infty.$$

Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(Z_n^{(\beta=1)}) \quad \xrightarrow{\text{dist.}} \quad \sqrt{2}\sigma\xi^{\mathbb{C}}.$$

Key lemma (quite complicated proof).

There exists a universal constant K such that

$$\mathbb{E}\left[|p_m(Z_n^{(\beta=1)})|^2\right] \le Km \qquad \text{for all positive integers } m, n.$$

Linear combination of traces ($\beta = 4$)

Theorem 4. [Jiang–M (2014)] CLT for CSE

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers satisfying $\sum_{j=1}^{\infty} (j \log j)|a_j|^2 < \infty$. Set $\sigma^2 = \sum_{j=1}^{\infty} j|a_j|^2$. Then, as $n \to \infty$,

$$\sum_{j=1}^{\infty} a_j p_j(Z_n^{(\beta=4)}) \xrightarrow{\text{dist.}} \frac{1}{\sqrt{2}} \sigma \xi^{\mathbb{C}}.$$

Key lemma (quite complicated proof).

There exists a universal constant K such that

$$\mathbb{E}\left[|p_m(Z_n^{(\beta=4)})|^2\right] \le K m \log m \qquad \text{for all positive integers } m, n.$$

Remarks and Future works

1. ($\beta = 4$) It does not seem that we can remove $\log m$ in the evaluation $\mathbb{E}\left[|p_m(Z_n^{(\beta=4)})|^2\right] \leq Km \log m$. In fact, we proved that there exist universal constants K_1, K_2 such that (if $n = m$) $K_1 m \log m \leq \mathbb{E}\left[|p_m(Z_m^{(\beta=4)})|^2\right] \leq K_2 m \log m$.
2. (general $\beta > 0$) Obtain the CLT for $\sum_{j=1}^{\infty} a_j p_j(Z_n^{(\beta)})$ ($\to \sqrt{\frac{2}{\beta}}\sigma\xi^{\mathbb{C}}$).
3. (general $\beta > 0$) Consider the convergence rate. Kurt Johansson (1997) obtained the CUE case.
4. Consider a Macdonald version.
(Jack polynomial $\xrightarrow{generalization}$ Macdonald polynomial)

1. Introduction

2. Diaconis–Shahshahani Theorem for CUE

3. Moment inequalities for traces

4. CLT for COE and CSE