コンパクト対称空間に付随する行列空間上の多項式積分について (On polynomial integrals over matrix spaces associated with compact symmetric spaces)

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Introduction

Let $\mathcal{S}$ be a space of matrices $X = (x_{ij})$ equipped with a probability measure $\mathrm{d}X$. In other words, suppose that X is an $\mathcal{S}$-valued random matrix with distribution $\mathrm{d}X$.

Example: $\mathcal{S}$ is the real orthogonal group $\mathrm{O}(n)$ and $\mathrm{d}X$ is its normalized Haar measure.

Question

Compute integrals of monomials in matrix entries

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k}] = \int_{\mathcal{S}} x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_k j_k} \mathrm{d}X.$$

The case for unitary groups $\mathrm{U}(n)$ was studied by Benoît Collins (Tohoku) in 2003. He called his computation method Weingarten calculus after the work of Don Weingarten (1978).

A toy model

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}.$$

For non-negative integers a, b, c, d,

$$\begin{aligned} &\int_{\mathrm{SO}(2)} u_{11}^a u_{12}^b u_{21}^c u_{22}^d \, \mathrm{d}U \\ =&(-1)^b \int_0^{2\pi} \cos^{a+d}\theta \sin^{b+c}\theta \, \frac{\mathrm{d}\theta}{2\pi} \\ =&\begin{cases} (-1)^b \frac{(a+d-1)!! \; (b+c-1)!!}{(a+b+c+d)!!} & \text{if both } a+d \text{ and } b+c \text{ are even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Unitary group

$$\mathrm{U}(n) = \{U \in \mathrm{GL}(n, \mathbb{C}) \mid UU^* = I_n\}.$$

Proposition.

Let $U = (u_{ij})_{1 \le i,j \le n}$ be an $n \times n$ Haar-distributed unitary matrix. Let $k \neq l$. For four sequences $\mathbf{i} = (i_1, \dots, i_k)$, $\mathbf{j} = (j_1, \dots, j_k)$, $\mathbf{i}' = (i'_1, \dots, i'_l)$, $\mathbf{j} = (j'_1, \dots, j'_l)$,

$$\mathbb{E}[u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_k j_k} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_l j'_l}}] = 0.$$

Example:

$$\mathbb{E}[u_{11} u_{12}^2 u_{23} \overline{u_{11}^2 u_{12}}] = 0. \qquad \mathbb{E}[u_{11}^2 u_{12}^2 u_{31}] = 0.$$

Unitary group

Theorem. [Samuel ('80)], [Collins ('03)]

Let $U = (u_{ij})_{1 \le i,j \le n}$ be an $n \times n$ Haar-distributed unitary matrix. For four sequences $\mathbf{i} = (i_1, \dots, i_k)$, $\mathbf{j} = (j_1, \dots, j_k)$, $\mathbf{i}' = (i'_1, \dots, i'_k)$, $\mathbf{j} = (j'_1, \dots, j'_k)$, we have

$$\begin{aligned} &\mathbb{E}[u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_k j_k} \overline{u_{i'_1 j'_1} u_{i'_2 j'_2} \cdots u_{i'_k j'_k}}] \\ &= \sum_{\sigma \in S_k} \sum_{\tau \in S_k} \delta_\sigma(\mathbf{i}, \mathbf{i}') \delta_\tau(\mathbf{j}, \mathbf{j}') \mathrm{Wg}^{\mathrm{U}}(\sigma^{-1}\tau; n). \end{aligned}$$

Here

$$\delta_\sigma(\mathbf{i}, \mathbf{i}') = \begin{cases} 1 & \text{if } (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(k)}) = (i'_1, i'_2, \dots, i'_k), \\ 0 & \text{otherwise.} \end{cases}$$

Unitary group

The unitary Weingarten function is a class function on S_k. It is defined by

$$\mathrm{Wg}^{\mathrm{U}}(\sigma; n) = \frac{1}{k!} \sum_{\lambda \vdash k} \frac{f^\lambda}{\prod_{(i,j)\in\lambda}(n+j-i)} \chi^\lambda(\sigma) \qquad (\sigma \in S_k)$$

summed over all partitions λ of k.
Here
χ^λ: irreducible character of S_k associated with λ,
f^λ: degree of the irreducible representation associated with λ.

Example ($k = 2$):

$$\mathrm{Wg}^{\mathrm{U}}(\mathrm{id}_2; n) = \frac{1}{(n+1)(n-1)}. \qquad \mathrm{Wg}^{\mathrm{U}}((1\ 2); n) = \frac{-1}{n(n+1)(n-1)}.$$

Orthogonal group

$$\mathrm{O}(n) = \{R \in \mathrm{GL}(n, \mathbb{R}) \mid RR^{\mathrm{T}} = I_n\}.$$

Proposition.

Let $R = (r_{ij})_{1 \le i,j \le n}$ be an $n \times n$ Haar-distributed orthogonal matrix. If k is odd, then

$$\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_k j_k}] = 0.$$

Example:

$$\mathbb{E}[r_{11} r_{12}^2] = 0.$$

Orthogonal group

The hyperoctahedral group H_k is the subgroup of S_{2k} generated by

$$(2i-1\ 2i) \quad (i=1,2,\dots,k), \qquad (2i-1\ 2j-1)(2i\ 2j) \quad (1 \le i < j \le k).$$

Note: $H_k \simeq \{\pm 1\} \wr S_k$, $|H_k| = 2^k k!$.

The pair (S_{2k}, H_k) is a Gelfand pair, i.e, the algebra

$$L(S_{2k}, H_k) = \{f \in L(S_{2k}) \mid f(\zeta\sigma\zeta') = f(\sigma) \ (\sigma \in S_{2k},\ \zeta, \zeta' \in H_k)\}$$

is commutative with respect to the convolution product

$$(f_1 * f_2)(\sigma) = \sum_{\tau \in S_{2k}} f_1(\tau) f_2(\tau^{-1}\sigma).$$

Orthogonal group

The zonal spherical functions ω^λ, where $\lambda \vdash k$, form a linear basis of $L(S_{2k}, H_k)$. Here

$$\omega^\lambda(\sigma) = \frac{1}{2^k k!} \sum_{\zeta \in H_k} \chi^{2\lambda}(\sigma\zeta) \qquad (\sigma \in S_{2k})$$

with $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$.

Define the subset M_{2k} in S_{2k} by

$$M_{2k} = \left\{ \begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(2k-1) & \sigma(2k) \end{pmatrix} \;\middle|\; \begin{matrix} \sigma(2i-1)<\sigma(2i) \ (i=1,2,\dots,k), \\ \sigma(1)<\sigma(3)<\cdots<\sigma(2k-1) \end{matrix} \right\}.$$

Then $M_{2k} \simeq S_{2k}/H_k$.

Orthogonal group

Theorem. [Collins-Śniady ('06)], [Collins-M ('09)]

Let $R = (r_{ij})_{1 \le i,j \le n}$ be an $n \times n$ Haar-distributed orthogonal matrix. For two sequences $\mathbf{i} = (i_1, \dots, i_{2k})$, $\mathbf{j} = (j_1, \dots, j_{2k})$, we have

$$\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_{2k} j_{2k}}] = \sum_{\sigma \in M_{2k}} \sum_{\tau \in M_{2k}} \Delta_\sigma(\mathbf{i}) \Delta_\tau(\mathbf{j}) \mathrm{Wg}^{\mathrm{O}}(\sigma^{-1}\tau; n).$$

Here

$$\Delta_\sigma(\mathbf{i}) = \begin{cases} 1 & \text{if } i_{\sigma(2r-1)} = i_{\sigma(2r)} \text{ for all } r, \\ 0 & \text{otherwise.} \end{cases}$$

Orthogonal group

The orthogonal Weingarten function belongs to $L(S_{2k}, H_k)$ and is given by

$$\mathrm{Wg}^{\mathrm{O}}(\sigma; n) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{\prod_{(i,j)\in\lambda}(n+2j-i-1)} \omega^\lambda(\sigma) \quad (\sigma \in S_{2k}).$$

Example ($k = 2$):

$$\mathrm{Wg}^{\mathrm{O}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right); n\right) = \frac{n+1}{n(n+2)(n-1)}.$$

$$\mathrm{Wg}^{\mathrm{O}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right); n\right) = \frac{-1}{n(n+2)(n-1)}.$$

Orthogonal group

$$\mathbb{E}[r_{ij}^{2k}] = \sum_{\sigma \in M_{2k}} \sum_{\tau \in M_{2k}} \mathrm{Wg}^{\mathrm{O}}(\sigma^{-1}\tau; n) = \frac{(2k-1)!!}{n(n+2)\cdots(n+2k-2)}.$$

$$\begin{aligned}\mathbb{E}[r_{11}r_{12}r_{21}r_{22}] &= \sum_{\sigma \in M_4} \sum_{\tau \in M_4} \Delta_\sigma((1,1,2,2))\Delta_\tau((1,2,1,2))\mathrm{Wg}^{\mathrm{O}}(\sigma^{-1}\tau; n) \\ &= \mathrm{Wg}^{\mathrm{O}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right); n\right) \\ &= \mathrm{Wg}^{\mathrm{O}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right); n\right) = \frac{-1}{n(n+2)(n-1)}.\end{aligned}$$

Symplectic group

$$\mathrm{Sp}(n) = \{S \in \mathrm{U}(2n) \mid SS^{\mathrm{D}} = I_{2n}\}, \quad S^{\mathrm{D}} := JS^{\mathrm{T}}J^{\mathrm{T}},$$
$$J = J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad \langle v, w \rangle := v^{\mathrm{T}} J w \ (v, w \in \mathbb{C}^{2n}).$$

Let $\epsilon = \mathrm{sgn}|_{H_k}$. The triple (S_{2k}, H_k, ϵ) is a twisted Gelfand pair, i.e., the algebra

$$L^{\epsilon}(S_{2k}, H_k) = \{f \in L(S_{2k}) \mid f(\zeta\sigma\zeta') = \epsilon(\zeta)\epsilon(\zeta')f(\sigma) \ (\sigma \in S_{2k}, \ \zeta, \zeta' \in H_k)\}$$

is commutative with respect to the convolution product.
The twisted zonal spherical functions π^{λ}, where $\lambda \vdash k$, form a linear basis of $L^{\epsilon}(S_{2k}, H_k)$. Here

$$\pi^{\lambda}(\sigma) = (2^k k!)^{-1} \sum_{\zeta \in H_k} \epsilon(\zeta) \chi^{\lambda \cup \lambda}(\sigma\zeta) \qquad (\sigma \in S_{2k})$$

with $\lambda \cup \lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots)$.

Symplectic group

Theorem. [Collins-Śniady ('06)], [M ('13)]

Let $S = (s_{ij})_{1 \le i,j \le 2n}$ be a $2n \times 2n$ Haar-distributed symplectic matrix. For two sequences $\mathbf{i} = (i_1, \dots, i_{2k})$, $\mathbf{j} = (j_1, \dots, j_{2k})$, we have

$$\mathbb{E}[s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_{2k} j_{2k}}] = \sum_{\sigma \in M_{2k}} \sum_{\tau \in M_{2k}} \Delta'_\sigma(\mathbf{i}) \Delta'_\tau(\mathbf{j}) \mathrm{Wg}^{\mathrm{Sp}}(\sigma^{-1}\tau; n).$$

Here

$$\Delta'_\sigma(\mathbf{i}) = \prod_{r=1}^{k} \langle e_{i_{\sigma(2r-1)}}, e_{i_{\sigma(2r)}} \rangle \qquad \in \{1, 0, -1\}.$$

($e_1, \dots, e_{2n}$: canonical basis of $\mathbb{C}^{2n}$.)

Symplectic group

The symplectic Weingarten function belongs to $L^{\epsilon}(S_{2k}, H_k)$ and is given by

$$\mathrm{Wg}^{\mathrm{Sp}}(\sigma; n) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{\lambda \cup \lambda}}{\prod_{(i,j)\in\lambda}(2n - 2i + j + 1)} \pi^{\lambda}(\sigma) \quad (\sigma \in S_{2k}).$$

Example ($k = 2$):

$$\mathrm{Wg}^{\mathrm{Sp}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right); n\right) = \frac{2n-1}{4n(n-1)(2n+1)}.$$

$$\mathrm{Wg}^{\mathrm{Sp}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right); n\right) = \frac{1}{4n(n-1)(2n+1)}.$$

Cartan's classification

Class $\mathcal{C}$	Symmetric spaces	Random matrices
A I	$\mathrm{U}(n)/\mathrm{O}(n)$	circular orthogonal ensemble (COE)
A II	$\mathrm{U}(2n)/\mathrm{Sp}(n)$	circular symplectic ensemble (CSE)
A III	$\mathrm{U}(n)/\mathrm{U}(a) \times \mathrm{U}(b)$	
BD I	$\mathrm{O}(n)/\mathrm{O}(a) \times \mathrm{O}(b)$	chiral ensemble
C II	$\mathrm{Sp}(n)/\mathrm{Sp}(a) \times \mathrm{Sp}(b)$	
	$(n = a + b)$	
D III	$\mathrm{O}(2n)/\mathrm{U}(n)$	Bogoliubov-de Gennes (BdG) ensemble
C I	$\mathrm{Sp}(n)/\mathrm{U}(n)$	

Figure: Classical compact symmetric spaces

Matrix realizations due to E. Dueñez ('04)

Let G/K be a compact symmetric space in the previous list.
$\Omega : G \to G$: the involution of G whose fix-point set is K.

$$\mathcal{S} := \{g\Omega(g)^{-1} \mid g \in G\} \quad \subset G.$$

G acts on $\mathcal{S}$: $\quad g_0.V = g_0 V \Omega(g_0)^{-1} \ \ (g_0 \in G,\ V \in \mathcal{S})$.
Then we can identify

$$G/K \simeq \mathcal{S}; \qquad G \ni g \mapsto g\Omega(g)^{-1}.$$

This bijection induces a probability measure $\mathrm{d}V$ on $\mathcal{S}$, which is invariant under the action of G. The probability space $(\mathcal{S}, \textbf{Borel}, \mathrm{d}V)$ is called the matrix ensemble associated with G/K (or with class $\mathcal{C}$).

Class A I: circular orthogonal ensemble (COE)

$$G/K = \mathrm{U}(n)/\mathrm{O}(n), \quad \Omega(g) = (g^{\mathrm{T}})^{-1},$$

$$\mathcal{S}^{\mathrm{A\,I}}(n) = \{V \in \mathrm{U}(n) \mid V \text{ is symmetric}\}.$$

Theorem [Brouwer-Beenakker ('96)], [M ('12)].

Let $V = V^{\mathrm{A\,I}} = (v_{ij})_{1 \le i,j \le n}$ be an $n \times n$ COE matrix. For two sequences $\mathbf{i} = (i_1, \dots, i_{2k})$ and $\mathbf{j} = (j_1, \dots, j_{2k})$, we have

$$\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2k-1} i_{2k}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2k-1} j_{2k}}}] = \sum_{\sigma \in S_{2k}} \delta_\sigma(\mathbf{i}, \mathbf{j}) \mathrm{Wg}^{\mathrm{A\,I}}(\sigma; n)$$

with

$$\mathrm{Wg}^{\mathrm{A\,I}}(\sigma; n) = \mathrm{Wg}^{\mathrm{O}}(\sigma; n+1) \qquad (\sigma \in S_{2k}).$$

If $k \neq l$ then $\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2k-1} i_{2k}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2l-1} j_{2l}}}] = 0.$

Class A I: circular orthogonal ensemble (COE)

Corollary [M ('12)]

Let $V = V^{\mathrm{A\,I}} = (v_{ij})$ be an $n \times n$ COE matrix. Let k be a positive integer.

$$\mathbb{E}[|v_{ii}|^{2k}] = \frac{2^k k!}{(n+1)(n+3)\cdots(n+2k-1)},$$

$$\mathbb{E}[|v_{ij}|^{2k}] = \frac{k!}{n(n+1)(n+2)\cdots(n+k-2)(n+2k-1)} \quad (i \neq j).$$

Class A II: circular symplectic ensemble (CSE)

$G/K = \mathrm{U}(2n)/\mathrm{Sp}(n), \quad \Omega(g) = (g^{\mathrm{D}})^{-1},$

$$\mathcal{S}^{\mathrm{A\,II}}(n) = \{V \in \mathrm{U}(2n) \mid V^{\mathrm{D}} = V\}.$$

Theorem [M ('13)]

Let $V = V^{\mathrm{A\,II}}$ be a $2n \times 2n$ CSE matrix and put $\hat{V} = (\hat{v}_{ij}) := VJ$. For two sequences $\mathbf{i} = (i_1, \dots, i_{2k})$ and $\mathbf{j} = (j_1, \dots, j_{2k})$, we have

$$\mathbb{E}[\hat{v}_{i_1 i_2} \hat{v}_{i_3 i_4} \cdots \hat{v}_{i_{2k-1} i_{2k}} \overline{\hat{v}_{j_1 j_2} \hat{v}_{j_3 j_4} \cdots \hat{v}_{j_{2k-1} j_{2k}}}] = \sum_{\sigma \in S_{2k}} \delta_\sigma(\mathbf{i}, \mathbf{j}) \mathrm{Wg}^{\mathrm{A\,II}}(\sigma; n)$$

with

$$\mathrm{Wg}^{\mathrm{A\,II}}(\sigma; n) = \mathrm{Wg}^{\mathrm{Sp}}(\sigma; n - \tfrac{1}{2}) \qquad (\sigma \in S_{2k}).$$

If $k \neq l$ then $\mathbb{E}[\hat{v}_{i_1 i_2} \hat{v}_{i_3 i_4} \cdots \hat{v}_{i_{2k-1} i_{2k}} \overline{\hat{v}_{j_1 j_2} \hat{v}_{j_3 j_4} \cdots \hat{v}_{j_{2l-1} j_{2l}}}] = 0.$

Example

Let $V = V^{\mathrm{A\,II}}$ be a $2n \times 2n$ CSE matrix. Consider

$$\mathbb{E}[|\hat{v}_{12}\hat{v}_{23}|^2] = \mathbb{E}[\hat{v}_{12}\hat{v}_{23}\overline{\hat{v}_{12}\hat{v}_{23}}] \qquad (\mathbf{i} = \mathbf{j} = (1,2,2,3),\ k = 2).$$

We observe that $\delta_\sigma(\mathbf{i},\mathbf{j}) = 1$ if and only if $\sigma \in \{\left(\begin{smallmatrix}1&2&3&4\\1&2&3&4\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right)\}$. Hence

$$\begin{aligned}\mathbb{E}[|\hat{v}_{12}\hat{v}_{23}|^2] =& \mathrm{Wg}^{\mathrm{A\,II}}(\left(\begin{smallmatrix}1&2&3&4\\1&2&3&4\end{smallmatrix}\right); n) + \mathrm{Wg}^{\mathrm{A\,II}}(\left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right); n) \\ =& \frac{n-1}{n(2n-1)(2n-3)} + (-1)\frac{1}{2n(2n-1)(2n-3)} \\ =& \frac{1}{2n(2n-1)}.\end{aligned}$$

Class A III

$$G/K = \mathrm{U}(n)/(\mathrm{U}(a) \times \mathrm{U}(b)), \quad n = a + b, \quad \Omega(g) = I'_{ab} g I'_{ab},$$
$$I' = I'_{ab} = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}.$$

$\mathcal{S}^{\mathrm{A\,III}}(a,b) = \{WI' \mid W$ is unitary and Hermitian with signature $(a,b)\}$.

Theorem [M ('13)]

Let $W = W^{\mathrm{A\,III}}$ be a $n \times n$ random matrix associated with class A III. For two sequences $\mathbf{i} = (i_1, \dots, i_k)$ and $\mathbf{j} = (j_1, \dots, j_k)$, we have

$$\mathbb{E}[w_{i_1 j_1} w_{i_2 j_2} \cdots w_{i_k j_k}] = \sum_{\sigma \in S_k} \delta_\sigma(\mathbf{i}, \mathbf{j}) \mathrm{Wg}^{\mathrm{A\,III}}(\sigma; a, b)$$

where $\mathrm{Wg}^{\mathrm{A\,III}}(\cdot; a, b)$ is a class function on S_k.

Class A III

Furthermore, the A III Weingarten function $\mathrm{Wg}^{\mathrm{A\,III}}(\cdot; a, b)$ can be expanded in terms of irreducible characters χ^λ of S_k as follows.

$$\mathrm{Wg}^{\mathrm{A\,III}}(\sigma; a, b) = \frac{1}{k!} \sum_{\lambda \vdash k} f^\lambda \frac{s_\lambda(1^a, (-1)^b)}{s_\lambda(1^n)} \chi^\lambda(\sigma) \qquad (\sigma \in S_k).$$

Here $s_\lambda(x_1, \dots, x_n)$ is the Schur polynomial.

Example ($k = 2$):

$$\mathrm{Wg}^{\mathrm{A\,III}}(\left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right); a, b) = \frac{(a-b+1)(a-b-1)}{(n+1)(n-1)}.$$

$$\mathrm{Wg}^{\mathrm{A\,III}}(\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right); a, b) = \frac{4ab}{n(n+1)(n-1)}.$$

Class A III

Example.

$$\mathbb{E}[w_{ij}^k] = \delta_{ij} \frac{k! \cdot s_{(k)}(1^a, (-1)^b)}{n(n+1)\cdots(n+k-1)}.$$

Here the numerators for k small are given as follows.

$$\begin{aligned} s_{(1)}(1^a, (-1)^b) =& a - b, \\ 2! \cdot s_{(2)}(1^a, (-1)^b) =& n + (a-b)^2, \end{aligned}$$

Class BD I

$G/K = \mathrm{O}(n)/(\mathrm{O}(a) \times \mathrm{O}(b)), \quad n = a + b, \quad \Omega(g) = I'_{ab} g I'_{ab},$
$I' = I'_{ab} = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}.$

$\mathcal{S}^{\mathrm{BD\,I}}(a, b) = \{WI' \mid W$ is orthogonal and symmetric with signature $(a, b)\}$.

Theorem [M ('13)]

Let $W = W^{\mathrm{BD\,I}}$ be a $n \times n$ random matrix associated with class BD I. For a sequence $\mathbf{i} = (i_1, \dots, i_{2k})$, we have

$$\mathbb{E}[w_{i_1 i_2} w_{i_3 i_4} \cdots w_{i_{2k-1} i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_\sigma(\mathbf{i}) \mathrm{Wg}^{\mathrm{BD\,I}}(\sigma; a, b)$$

where $\mathrm{Wg}^{\mathrm{BD\,I}}(\cdot; a, b)$ belongs to $L(S_{2k}, H_k)$.

Class BD I

Furthermore, the BD I Weingarten function $\mathrm{Wg}^{\mathrm{BD\,I}}(\cdot; a, b)$ can be expanded in terms of zonal characters ω^λ of the Gelfand pair (S_{2k}, H_k) as follows.

$$\mathrm{Wg}^{\mathrm{BD\,I}}(\sigma; a, b) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f^{2\lambda} \frac{Z_\lambda(1^a, (-1)^b)}{Z_\lambda(1^n)} \omega^\lambda(\sigma) \qquad (\sigma \in S_{2k}).$$

Here $Z_\lambda(x_1, \dots, x_n)$ is the zonal polynomial.

Example ($k = 2$):

$$\mathrm{Wg}^{\mathrm{BD\,I}}(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right); a, b) = \frac{(a-b)^2(n+1) - 2n}{n(n+2)(n-1)}.$$

$$\mathrm{Wg}^{\mathrm{BD\,I}}(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right); a, b) = \frac{4ab}{n(n+2)(n-1)}.$$

Class C II

$$G/K = \mathrm{Sp}(n)/(\mathrm{Sp}(a) \times \mathrm{Sp}(b)), \quad n = a + b, \quad \Omega(g) = I''_{ab} g I''_{ab},$$
$$I'' = I''_{ab} = \begin{pmatrix} I'_a & 0 \\ 0 & I'_b \end{pmatrix}.$$

$$\mathcal{S}^{\mathrm{C\,II}}(a,b) = \{WI'' \mid W \in \mathrm{Sp}(n) \text{ is Hermitian with signature } (a,b)\}.$$

Theorem [M ('13)]

Let $W = W^{\mathrm{C\,II}}$ be a $2n \times 2n$ random matrix associated with class C II and put $\hat{W} := WJ$. For a sequence $\mathbf{i} = (i_1, \dots, i_{2k})$, we have

$$\mathbb{E}[\hat{w}_{i_1 i_2} \hat{w}_{i_3 i_4} \cdots \hat{w}_{i_{2k-1} i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta'_\sigma(\mathbf{i}) \mathrm{Wg}^{\mathrm{C\,II}}(\sigma; a, b)$$

where $\mathrm{Wg}^{\mathrm{C\,II}}(\cdot; a, b)$ belongs to $L^\epsilon(S_{2k}, H_k)$.

Class C II

Furthermore, the C II Weingarten function $\mathrm{Wg}^{\mathrm{C\,II}}(\cdot; a, b)$ can be expanded in terms of twisted zonal characters π^λ of the twisted Gelfand pair (S_{2k}, H_k, ϵ) as follows.

$$\mathrm{Wg}^{\mathrm{C\,II}}(\sigma; a, b) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f^{\lambda \cup \lambda} \frac{Z'_\lambda(1^a, (-1)^b)}{Z'_\lambda(1^n)} \pi^\lambda(\sigma) \qquad (\sigma \in S_{2k}).$$

Here $Z'_\lambda(x_1, \dots, x_n)$ is the symplectic zonal polynomial.

Example ($k = 2$):

$$\mathrm{Wg}^{\mathrm{C\,II}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right); a, b\right) = \frac{(a-b)^2(2n-1) - n}{n(2n+1)(n-1)}.$$

$$\mathrm{Wg}^{\mathrm{C\,II}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right); a, b\right) = \frac{4ab}{n(2n+1)(n-1)}.$$

Class D III

$$G = \mathrm{O}(2n), \ \ \Omega(g) = (g^{\mathrm{D}})^{-1} = JgJ^{\mathrm{T}}, \ \ K = \mathrm{O}(2n) \cap \mathrm{Sp}(n) \cong \mathrm{U}(n).$$

$$\mathcal{S}^{\mathrm{D\,III}}(n) = \{ V \in \mathrm{O}(2n) \mid VJ \text{ is "dexter" antisymmetric } \}.$$

Theorem [M ('13)]

Let $V = V^{\mathrm{D\,III}}$ be a $2n \times 2n$ random matrix associated with class D III and put $\hat{V} := VJ$. For a sequence $\mathbf{i} = (i_1, \dots, i_{2k})$ with an even integer k, we have

$$\mathbb{E}[\hat{v}_{i_1 i_2} \hat{v}_{i_3 i_4} \cdots \hat{v}_{i_{2k-1} i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_\sigma(\mathbf{i}) \mathrm{Wg}^{\mathrm{D\,III}}(\sigma; n).$$

If k is odd, then $\mathbb{E}[\hat{v}_{i_1 i_2} \hat{v}_{i_3 i_4} \cdots \hat{v}_{i_{2k-1} i_{2k}}] = 0$.

Class D III

Suppose that k is even. Define the function $T_n^{\mathrm{D\,III}}$ on S_{2k} by

$$T_n^{\mathrm{D\,III}}(\zeta\sigma\zeta') = \epsilon(\zeta)\, T_n^{\mathrm{D\,III}}(\sigma) \qquad (\sigma \in S_{2k},\ \zeta, \zeta' \in H_k); \qquad (\spadesuit)$$

$$T_n^{\mathrm{D\,III}}(\sigma_\mu) = \begin{cases} (-1)^{\frac{k}{2}}(2n)^{\ell(\mu)} & \text{if } \mu \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\sigma_\mu \in S_{2k}$ is the "typical" permutation of coset-type μ.

Definition of the D III Weingarten function

$$\mathrm{Wg}^{\mathrm{D\,III}}(\cdot; n) = \frac{1}{2^k k!} \cdot T_n^{\mathrm{D\,III}} * \mathrm{Wg}^{\mathrm{O}}(\cdot; 2n).$$

This satisfies ♠.

Example. $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right) = (3\ 4)\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right)$.

$$\mathrm{Wg}^{\mathrm{D\,III}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right); n\right) = -\mathrm{Wg}^{\mathrm{D\,III}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right); n\right) = \frac{1}{2n-1}.$$

Class C I

$$G = \mathrm{Sp}(n), \quad \Omega(g) = I'_{nn} g I'_{nn}, \quad K = \{ \left(\begin{smallmatrix} U & 0 \\ 0 & \overline{U} \end{smallmatrix}\right) \mid U \in \mathrm{U}(n) \} \cong \mathrm{U}(n).$$

$$\mathcal{S}^{\mathrm{CI}}(n) = \{ V = W I'_{nn} \mid W \text{ is unitary and Hermitian with } JW = -\overline{W} J \}.$$

Theorem [M ('13)]

Let $W = W^{\mathrm{CI}}$ be a $2n \times 2n$ random matrix associated with class C I and put $\hat{W} := WJ$. For a sequence $\mathbf{i} = (i_1, \dots, i_{2k})$ with an even integer k, we have

$$\mathbb{E}[\hat{w}_{i_1 i_2} \hat{w}_{i_3 i_4} \cdots \hat{w}_{i_{2k-1} i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta'_\sigma(\mathbf{i}) \mathrm{Wg}^{\mathrm{CI}}(\sigma; n).$$

If k is odd, then $\mathbb{E}[\hat{w}_{i_1 i_2} \hat{w}_{i_3 i_4} \cdots \hat{w}_{i_{2k-1} i_{2k}}] = 0$.

Class C I

Suppose that k is even. Define the function $T_n^{\mathrm{C\,I}}$ on S_{2k} by

$$T_n^{\mathrm{C\,I}}(\zeta\sigma\zeta') = \epsilon(\zeta')\,T_n^{\mathrm{C\,I}}(\sigma) \qquad (\sigma \in S_{2k},\ \zeta, \zeta' \in H_k); \qquad (\clubsuit)$$

$$T_n^{\mathrm{C\,I}}(\sigma_\mu) = \begin{cases} (-2n)^{\ell(\mu)} & \text{if } \mu \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition of the C I Weingarten function

$$\mathrm{Wg}^{\mathrm{C\,I}}(\cdot; n) = \frac{1}{2^k k!} \cdot T_n^{\mathrm{C\,I}} * \mathrm{Wg}^{\mathrm{Sp}}(\cdot; n).$$

This satisfies $\clubsuit$.

Example. $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right) = (3\ 4)\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right)$.

$$\mathrm{Wg}^{\mathrm{C\,I}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right); n\right) = \mathrm{Wg}^{\mathrm{C\,I}}\left(\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{smallmatrix}\right); n\right) = \frac{-1}{2n+1}.$$

Conclusion

Integrals of monomials over random matrix ensembles associated compact symmetric spaces are of the form

$$\sum_{\sigma} (\text{a delta function in } \sigma) \times \mathrm{Wg}^{\mathcal{C}}(\sigma; n).$$

Here the Weingarten function are:

- class functions on S_k if $\mathcal{C} =$ A, A III;
- H_k-biinvariant functions on S_{2k} if $\mathcal{C} =$ B/D, A I, BD I;
- H_k-bi-twisted functions on S_{2k} if $\mathcal{C} =$ C, A II, C II.
- H_k-invariant (one side) and H_k-twisted (another side) functions on S_{2k} if $\mathcal{C} =$ D III, C I.

Remarks

- We have established Weingarten calculus for $\mathrm{U}(n), \mathrm{O}(n), \mathrm{Sp}(n)$.
- However, NOT for $\mathrm{SU}(n), \mathrm{SO}(n)$ (in process).
  - Since $\mathrm{SU}(2) = \mathrm{Sp}(1)$, we obtain

$$\int_{\mathrm{SU}(2)} u_{11} u_{12} u_{21} u_{22} \, \mathrm{d}U = -\frac{1}{6}.$$

  - For any $\sigma \in S_n$,

$$\int_{\mathrm{SU}(n)} u_{1\sigma(1)} u_{2\sigma(2)} \cdots u_{n\sigma(n)} \, \mathrm{d}U = \frac{\mathrm{sgn}(\sigma)}{n!}.$$

    Using this fact, we can see that

$$\int_{\mathrm{SU}(n)} (\det U) \, \mathrm{d}U = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \int_{\mathrm{SU}(n)} u_{1\sigma(1)} u_{2\sigma(2)} \cdots u_{n\sigma(n)} \, \mathrm{d}U$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \frac{\mathrm{sgn}(\sigma)}{n!} = 1.$$

Remarks

- **Weingarten calculus for the compact exceptional Lie group G_2.** The compact Lie group G_2 can be embedded into $\mathrm{SO}(V)$, where V is the 7-dimensional irreducible representation and consists of traceless octonions.

$$\int_{G_2} g_{11}g_{44}g_{55}\,\mathrm{d}g = \frac{1}{42}, \quad \int_{G_2} g_{11}g_{33}g_{77}\,\mathrm{d}g = 0,$$

$$\int_{G_2} g_{11}g_{14}g_{33}g_{42}g_{74}\,\mathrm{d}g = \frac{1}{1512}, \quad \int_{G_2} g_{11}g_{14}g_{37}g_{42}g_{74}\,\mathrm{d}g = 0.$$

(in process)

Remarks

- **Asymptotoic behavior and Jucys-Murphy elements.**

$$\mathrm{Wg}(\sigma; n) = \sum_{r=0}^{\infty} a_r(\sigma) n^{-r}.$$

  The coefficients $a_r(\sigma)$ have combinatorial interpretations related to Jucys-Murphy elements.
  [M-Novak ('13)], [M ('11)], [Féray ('12)].

- **Applications**: Weingarten calculus is applied to Harish-Chandra integrals, mathematical physics, statistics, quantum information theory, designs....

Thank you for listening

Reference

1. Random Matrices: Theory and Applications 1 (2012), 1250005, 18 pages.
2. Random Matrices: Theory and Applications 2 (2013), 1350001, 26 pages.