ガウス型べき級数とパフィアンの恒等式

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白井朋之氏（九州大 IMI）との共同研究

次のランダムべき級数を考える.

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

ここで，係数 $\{a_k\}_{k=0}^{\infty}$ は独立同分布 (i.i.d.) で，実標準正規分布 $\mathrm{N}_{\mathbb{R}}(0,1)$ に従う.
収束半径は，ほとんど確実に 1 となる．さらに収束円 $|z| = 1$ は，ほとんど確実に自然境界となる.

問題. ランダム級数 f の零点の分布はどうなっているか？

関連する仕事(1)

Kac (1943). Kac 多項式:

$$p_n(z) := \sum_{k=0}^{n} a_k z^k, \qquad \{a_k\}_{k=0}^n\text{: i.i.d. } \mathrm{N}_{\mathbb{R}}(0,1).$$

$N_n(\Omega)$: $\mathbb{R}$ の可測集合 Ω に属する p_n の実零点の個数.

$$\mathbb{E}[N_n(\Omega)] = \frac{1}{\pi}\int_\Omega \sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}\,dt.$$

特に実零点の個数の期待値は, $\mathbb{E}[N_n(\mathbb{R})] \sim \frac{2}{\pi}\log n$ $(n \to \infty)$.

Shepp-Vanderbei (1995): 複素零点へ拡張.
Logan-Shepp (1968): 安定分布（パラメータ $0 < \alpha \leq 2$）へ拡張.
（$\alpha = 1$ がコーシー分布, $\alpha = 2$ が正規分布.）

関連する仕事 (2)

ランダム行列論. ランダム特性多項式:

$$\phi_X(\lambda) = \det(\lambda I - X),$$

ここで，X は $N \times N$ ランダム行列.

$$\phi_X \text{ の零点} \quad \leftrightarrow \quad X \text{ の固有値}.$$

X を GOE 行列とする．(GOE=Gaussian Orthogonal Ensemble.)

$$X = \frac{1}{2}(A + {}^t A), \quad A = (a_{ij}), \quad \{a_{ij}\}\text{: i.i.d. } \mathrm{N}_{\mathbb{R}}(0,1).$$

X は実対称行列なので，固有値は実数．固有値の同時密度関数は以下の形で与えられる．

$$C_N e^{-\sum \lambda_j^2/2} \prod_{i<j} |\lambda_i - \lambda_j|.$$

固有値の分布は，相関関数を用いて記述される．
$n = 1, 2, \ldots, N$. n 点相関関数:

$$
\begin{aligned}
&\rho_n(\lambda_1, \ldots, \lambda_n) \\
=&\text{“各 } \lambda_j \text{ に } X \text{ の固有値がある確率”} \\
=&\frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} C_N e^{-\sum_{j=1}^{N} \lambda_j^2/2} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j| \, d\lambda_{n+1} \cdots d\lambda_N.
\end{aligned}
$$

ρ_n はパフィアンで与えられることがよく知られている．
すなわち，ランダム特性多項式 ϕ_X の零点分布（GOE 行列 X の固有値分布）は，パフィアン点過程をなす．

関連する仕事 (3)

Peres-Virág (2005).

$$f_{\mathbb{C}}(z) := \sum_{k=0}^{\infty} \zeta_k z^k, \qquad \{\zeta_k\}_{k=0}^{\infty}\text{: i.i.d. } \mathrm{N}_{\mathbb{C}}(0,1).$$

$f_{\mathbb{C}}$ の零点分布の相関関数を考える:

$$\rho_n(z_1, \ldots, z_n) := \text{“各 } z_j \text{ に } f_{\mathbb{C}} \text{ の零点がある確率”}.$$

このとき,

$$\rho_n(z_1, \ldots, z_n) = \det \left(\frac{1}{(1 - z_i \bar{z}_j)^2} \right)_{1 \le i,j \le n}$$

すなわち, $f_{\mathbb{C}}$ の零点分布は行列式点過程をなす.

研究の動機

Peres-Virág (2005) の結果の「実数版」を考えたい．その場合，行列式の代わりにパフィアンが必要となることが期待される．

Peres-Virág (2005) の証明では，次の公式が重要な役割を果たした．
Borchardt (1855) :

$$\det\left(\frac{1}{1-x_iy_j}\right)_{n\times n}\cdot\operatorname{per}\left(\frac{1}{1-x_iy_j}\right)_{n\times n}=\det\left(\frac{1}{(1-x_iy_j)^2}\right)_{n\times n}$$

この恒等式の類似が既に得られている．石川-川向-岡田 (2005):

$$\operatorname{Pf}\left(\frac{x_i-x_j}{1-x_ix_j}\right)_{2n\times 2n}\cdot\operatorname{Hf}\left(\frac{1}{1-x_ix_j}\right)_{2n\times 2n}=\operatorname{Pf}\left(\frac{x_i-x_j}{(1-x_ix_j)^2}\right)_{2n\times 2n}.$$

パフィアンとハフニアン

S_{2n} を対称群とする.

$$F_n := \left\{ \sigma \in S_{2n} \;\middle|\; \begin{array}{l} \sigma(2i-1) < \sigma(2i) \ (i = 1, 2, \dots, n), \\ \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1) \end{array} \right\}$$

偶数次の交代行列 $B = (b_{ij})_{1 \le i,j \le 2n}$ のパフィアン:

$$\mathrm{Pf}\, B = \sum_{\sigma \in F_n} (\operatorname{sgn} \sigma) b_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)}.$$

偶数次の対称行列 $A = (a_{ij})_{1 \le i,j \le 2n}$ のハフニアン:

$$\mathrm{Hf}\, A = \sum_{\sigma \in F_n} a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2n-1)\sigma(2n)}.$$

$$\mathrm{Hf}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{12}. \qquad \mathrm{Pf}\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} = a_{12}.$$

$$\mathrm{Hf}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23}.$$

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

零点の相関

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \{a_k\}_{k=0}^{\infty}\text{: i.i.d. } \mathrm{N}_{\mathbb{R}}(0,1).$$

f の収束半径は，a.s. で 1．さらに円 $|z| = 1$ は a.s. で自然境界.

実零点と複素零点の **n 点相関関数**を，それぞれ

$$\rho_n^{\mathrm{r}}(t_1, \dots, t_n) \quad \text{and} \quad \rho_n^{\mathrm{c}}(z_1, \dots, z_n),$$

とおく．ここで，$t_1, t_2, \dots, t_n$ は区間 $(-1, +1)$ 内の実数で，
$z_1, z_2, \dots, z_n$ は $|z_i| < 1$ かつ $\mathrm{Im}\, z_i > 0$ なる複素数.

定理 (Real Zero Correlation Theorem).

$$\rho_n^{\mathrm{r}}(t_1,\dots,t_n) = \pi^{-n}\,\mathrm{Pf}(\mathbb{K}(t_i,t_j))_{1\le i,j\le n}.$$

ここで，各 $\mathbb{K}(s,t) = \begin{pmatrix} \mathbb{K}_{11}(s,t) & \mathbb{K}_{12}(s,t) \\ \mathbb{K}_{21}(s,t) & \mathbb{K}_{22}(s,t) \end{pmatrix}$ は 2×2 の行列で,

$$\begin{aligned}
\mathbb{K}_{11}(s,t) &= \frac{s-t}{\sqrt{(1-s^2)(1-t^2)}\,(1-st)^2},\\
\mathbb{K}_{12}(s,t) &= \sqrt{\frac{1-t^2}{1-s^2}}\frac{1}{1-st},\\
\mathbb{K}_{21}(s,t) &= -\sqrt{\frac{1-s^2}{1-t^2}}\frac{1}{1-st},\\
\mathbb{K}_{22}(s,t) &= \mathrm{sgn}(t-s)\arcsin\left(\frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}\right).
\end{aligned}$$

$$\mathbb{K}_{11}(s,t) = \frac{\partial^2}{\partial s\partial t}\mathbb{K}_{22}(s,t), \quad \mathbb{K}_{12}(s,t) = -\mathbb{K}_{21}(t,s) = \frac{\partial}{\partial s}\mathbb{K}_{22}(s,t).$$

定理 (Complex Zero Correlation Theorem).

$$\rho_n^{\mathrm{c}}(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^{n} \frac{1}{|1 - z_j^2|} \cdot \mathrm{Pf}(\mathbb{K}^{\mathrm{c}}(z_i, z_j))_{1 \le i,j \le n}.$$

ここで,

$$\mathbb{K}^{\mathrm{c}}(z, w) = \begin{pmatrix} \frac{z-w}{(1-zw)^2} & \frac{z-\bar{w}}{(1-z\bar{w})^2} \\ \frac{\bar{z}-w}{(1-\bar{z}w)^2} & \frac{\bar{z}-\bar{w}}{(1-\bar{z}\bar{w})^2} \end{pmatrix}.$$

したがって，f の実零点および複素零点の分布は，ともにパフィアン点過程をなす.

例 (1 点相関, 2 点相関)

$$\rho_1^{\mathrm{r}}(t) = \frac{1}{\pi(1-t^2)}. \qquad \rho_1^{\mathrm{c}}(z) = \frac{|z-\bar{z}|}{\pi|1-z^2|(1-|z|^2)^2}.$$

$$\rho_2^{\mathrm{r}}(t_1,t_2) = \frac{(t_1-t_2)^2}{\pi^2(1-t_1^2)(1-t_2^2)(1-t_1t_2)^2} + \frac{|t_1-t_2|}{\pi^2\sqrt{(1-t_1^2)(1-t_2^2)}(1-t_1t_2)^2} \arcsin\frac{\sqrt{(1-t_1^2)(1-t_2^2)}}{1-t_1t_2}.$$

$$\rho_2^{\mathrm{c}}(z_1,z_2) = \frac{1}{\pi^2|1-z_1^2||1-z_2^2|} \times \left[\frac{|z_1-\bar{z}_1|\,|z_2-\bar{z}_2|}{(1-|z_1|^2)^2(1-|z_2|^2)^2} + \frac{|z_1-z_2|^2}{|1-z_1z_2|^4} - \frac{|z_1-\bar{z}_2|^2}{|1-z_1\bar{z}_2|^4}\right].$$

Forrester による証明

Forrester (2010 arXiv) は，今の二つの定理を別の方法で独立に証明した．Forrester の証明のアイディアは次のようだった:

- f の零点分布は，あるランダム行列の固有値の，行列サイズを無限大にしたときの極限分布に一致する．[Krishnapur (2009)].
- そのランダム行列の固有値分布は，ランダム行列理論における標準的な手法（Forrester, 永尾，Borodin などにより発展）を用いて，具体的にパフィアンで表すことができる．

我々の証明はこれとは異なり，ランダム行列理論を経由せず，より直接的な証明である．先に述べた石川-川向-岡田によるパフィアン・ハフニアンの公式が鍵となる．

確率過程 $\{f(t)\}_{t\in(-1,+1)}$ は平均 0 の実ガウス過程をなす．すなわち，任意の $n \geq 1$ と $-1 < t_1 < t_2 < \cdots < t_n < 1$ に対して，$(f(t_1), f(t_2), \ldots, f(t_n))$ は平均 $\mathbf{0}$ の n 次元実ガウス分布 $\mathrm{N}_{\mathbb{R}}(\mathbf{0}, \Sigma(\mathbf{t}))$ に従う．ここで，共分散行列

$$\Sigma(\mathbf{t}) = (\mathbb{E}[f(t_i)f(t_j)])_{1\leq i,j\leq n}$$

は，

$$\mathbb{E}[f(s)f(t)] = \sum_{k,l} s^k t^l \mathbb{E}[a_k a_l] = \sum_k (st)^k = \frac{1}{1-st}$$

で与えられる．

定理 (Absolute Value Moment Theorem).

$$\mathbb{E}[|f(t_1)\cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2}\, \mathrm{Pf}(\mathbb{K}(t_i, t_j))_{1\leq i,j\leq n}.$$

ここで $\mathbb{K}(s,t)$ は，Real Zero Correlation Theorem で与えたものと同じ．

定理 (Sign Moment Theorem).

$$\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \prod_{1 \le i < j \le 2n} \operatorname{sgn}(t_j - t_i) \cdot \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1 \le i,j \le 2n}.$$

ここで $\mathbb{K}_{22}(s,t) = \operatorname{sgn}(t-s) \arcsin \frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}$.
したがって，特に $-1 < t_1 < \cdots < t_{2n} < 1$ ならば

$$\mathbb{E}[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = \operatorname{Pf}(\, \mathbb{E}[\operatorname{sgn} f(t_i) \operatorname{sgn} f(t_j)]\,)_{1 \le i < j \le 2n}.$$

n が奇数のときは，常に $\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_n)] = 0$.

例. $-1 < t_1 < t_2 < t_3 < t_4 < 1$ に対し，

$$\begin{aligned} &\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \operatorname{sgn} f(t_3) \operatorname{sgn} f(t_4)] \\ = &\qquad \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2)] \cdot \mathbb{E}[\operatorname{sgn} f(t_3) \operatorname{sgn} f(t_4)] \\ &\quad - \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_3)] \cdot \mathbb{E}[\operatorname{sgn} f(t_2) \operatorname{sgn} f(t_4)] \\ &\quad + \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_4)] \cdot \mathbb{E}[\operatorname{sgn} f(t_2) \operatorname{sgn} f(t_3)] \end{aligned}$$

であり，また

$$\mathbb{E}[\operatorname{sgn} f(t_i) \operatorname{sgn} f(t_j)] = \frac{2}{\pi} \arcsin \frac{\sqrt{(1-t_i^2)(1-t_j^2)}}{1-t_i t_j}.$$

Wick formula

Sign Moment Theorem は，Wick の公式に似ている．

Wick の公式.

$(X_1, X_2, \dots)$ が平均 **0** の実正規分布に従うとき，

$$\mathbb{E}[X_1 X_2 \cdots X_{2n}] = \mathrm{Hf}(\mathbb{E}[X_i X_j])_{1 \le i,j \le 2n}$$

が成り立つ．また m が奇数なら，常に $\mathbb{E}[X_1 \cdots X_m] = 0$.

$$\begin{aligned}\mathbb{E}[X_1 X_2 X_3 X_4] =& \mathbb{E}[X_1 X_2] \cdot \mathbb{E}[X_3 X_4] \\ &+ \mathbb{E}[X_1 X_3] \cdot \mathbb{E}[X_2 X_4] + \mathbb{E}[X_1 X_4] \cdot \mathbb{E}[X_2 X_3].\end{aligned}$$

$(X_1, X_2, \dots)$ を平均 **0** の実正規分布に従うとし，$\sigma_{ij} = \mathbb{E}[X_i X_j]$ とおく．このとき次の問いは自然である．

> 絶対値モーメント $\mathbb{E}[|X_1 \cdots X_n|]$ や符号モーメント $\mathbb{E}[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_n]$ に対して，Wick の公式の類似は存在するだろうか？

答えは否である．

符号モーメントに関しては，一般に n が奇数ならば $\mathbb{E}[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_n] = 0$ であり，また公式

$$\mathbb{E}[\operatorname{sgn} X_1 \operatorname{sgn} X_2] = \frac{2}{\pi} \arcsin \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}},$$

が知られている．ところが，$n \geq 4$ に関しては，いかなる式も知られていない．

絶対値モーメントについては次が知られている.

命題 [鍋谷 (1951, 1952)].

$$\mathbb{E}[|X_1X_2|] = \frac{2}{\pi}\left(\sqrt{\det\Sigma} + \sigma_{12}\arcsin\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right).$$

$$\mathbb{E}[|X_1X_2X_3|] = \left(\frac{2}{\pi}\right)^{3/2}\left(\sqrt{\det\Sigma}+\right.$$

$$\left.\sum_{(i,j,k)}\frac{\sigma_{ij}\sigma_{kk}+\sigma_{ik}\sigma_{jk}}{\sqrt{\sigma_{kk}}}\arcsin\frac{\sigma_{ij}\sigma_{kk}-\sigma_{ik}\sigma_{jk}}{\sqrt{(\sigma_{ii}\sigma_{kk}-\sigma_{ik}^2)(\sigma_{jj}\sigma_{kk}-\sigma_{jk}^2)}}\right).$$

ここで和は $(i,j,k) = (1,2,3),(2,3,1),(3,1,2)$ を走る.

4 次以上については，公式は知られていない.

このように，一般には $\mathbb{E}[|X_1 \cdots X_n|]$ や $\mathbb{E}[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_{2n}]$ は計算できない.
ところが我々の定理は，共分散が

$$\mathbb{E}[X_i X_j] = \frac{1}{1 - t_i t_j}$$

の形のときはパフィアンになると主張している！

Differential Sign Moment Formula

We first prove the formula for the derivation of the sign moment. Recall

$$\mathbb{K}_{11}(s,t) = \frac{s-t}{\sqrt{(1-s^2)(1-t^2)}(1-st)^2}.$$

Proposition (Differential Sign Moment Formula).

$$\begin{aligned}
&\frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\
&= \left(\tfrac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}_{11}(t_i,t_j))_{1 \le i,j \le 2n} \\
&= \left(\tfrac{2}{\pi}\right)^n \prod_{i=1}^{2n} \frac{1}{\sqrt{1-t_i^2}} \cdot \operatorname{Pf}\left(\frac{t_i - t_j}{(1-t_i t_j)^2}\right)_{1 \le i,j \le 2n}.
\end{aligned}$$

Proof of Differential Sign Moment Formula

Since $\frac{d}{dx}\operatorname{sgn} x = 2\delta_0(x)$, where δ_0 is the Dirac delta function, we have

$$
\begin{aligned}
&\frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\
&\mathbb{E}\left[\frac{\partial}{\partial t_1} \operatorname{sgn} f(t_1) \cdot \frac{\partial}{\partial t_2} \operatorname{sgn} f(t_2) \cdots \frac{\partial}{\partial t_{2n}} \operatorname{sgn} f(t_{2n})\right] \\
=&2^{2n}\mathbb{E}[\delta_0(f(t_1))f'(t_1)\cdots\delta_0(f(t_{2n}))f'(t_{2n})] \\
=&\frac{2^{2n}\mathbb{E}[f'(t_1)\cdots f'(t_{2n}) \mid f(t_1)=\cdots=f(t_{2n})=0]}{(2\pi)^n(\det\Sigma(\mathbf{t}))^{1/2}},
\end{aligned}
$$

where $\Sigma(\mathbf{t}) = ((1-t_it_j)^{-1})_{1\le i,j\le 2n}$.
Note that $(2\pi)^{-n}(\det\Sigma(\mathbf{t}))^{-1/2}$ is the density of the Gaussian vector $(f(t_1),\dots,f(t_{2n}))$ at $(0,\dots,0)$.

This heuristic discussion can be justified in the framework of Malliavin calculus.

Proof of Differential Sign Moment Formula

Lemma.

The conditional distribution of $(f'(t_1), \dots, f'(t_{2n}))$ given $f(t_1) = \cdots = f(t_{2n}) = 0$ has the same distribution with

$$(q_1(\mathbf{t})f(t_1), \dots, q_{2n}(\mathbf{t})f(t_{2n})).$$

Here

$$q_i(\mathbf{t}) = \frac{1}{1-t_i^2} \prod_{\substack{1 \le k \le 2n \\ k \ne i}} \frac{t_i - t_k}{1 - t_i t_k}.$$

Note:

$$(-1)^n q_1(\mathbf{t}) \cdots q_{2n}(\mathbf{t}) = \frac{\prod_{1 \le i < j \le 2n} (t_i - t_j)^2}{\prod_{i,j=1}^{2n} (1 - t_i t_j)} = \det \left(\frac{1}{1 - t_i t_j} \right)_{1 \le i,j \le 2n}.$$

Proof of Differential Sign Moment Formula

Hence,

$$\frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})]$$
$$=(-1)^n \left(\tfrac{2}{\pi}\right)^n (\det \Sigma(\mathbf{t}))^{1/2} \cdot \mathbb{E}[f(t_1) \cdots f(t_{2n})].$$

Cauchy's determinant formula and Wick formula give

$$= \left(\tfrac{2}{\pi}\right)^n \prod_{i=1}^{2n} \frac{1}{\sqrt{1-t_i^2}} \cdot \prod_{1 \le i < j \le 2n} \frac{t_i - t_j}{1 - t_i t_j} \cdot \operatorname{Hf}\left(\frac{1}{1 - t_i t_j}\right)_{1 \le i,j \le 2n}.$$

Recall Schur's pfaffian

$$\operatorname{Pf}\left(\frac{t_i - t_j}{1 - t_i t_j}\right)_{1 \le i,j \le 2n} = \prod_{1 \le i < j \le 2n} \frac{t_i - t_j}{1 - t_i t_j}.$$

Proof of Differential Sign Moment Formula

Lemma [Ishikawa-Kawamuko-Okada (2005)].

$$\mathrm{Pf}\left(\frac{t_i - t_j}{1 - t_i t_j}\right)_{i,j} \cdot \mathrm{Hf}\left(\frac{1}{1 - t_i t_j}\right)_{i,j} = \mathrm{Pf}\left(\frac{t_i - t_j}{(1 - t_i t_j)^2}\right)_{i,j}.$$

Hence we have

$$\frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})]$$
$$= \left(\tfrac{2}{\pi}\right)^n \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - t_i^2}} \cdot \mathrm{Pf}\left(\frac{t_i - t_j}{(1 - t_i t_j)^2}\right)_{1 \le i,j \le 2n}.$$

We thus finish the proof of the Differential Sign Moment Formula.

Proof of Sign Moment Theorem

Sign Moment Theorem.

For $-1 < t_1 < \cdots < t_{2n-1} < t_{2n} < 1$,

$$\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1 \le i,j \le 2n},$$

where $\mathbb{K}_{22}(s,t) = \operatorname{sgn}(t-s) \arcsin \frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}$.

Proof. Check

$$\frac{\partial^2}{\partial s \partial t} \mathbb{K}_{22}(s,t) = \mathbb{K}_{11}(s,t) = \frac{s-t}{\sqrt{(1-s^2)(1-t^2)}(1-st)^2}.$$

If we integrate Differential Sign Moment Formula with a good initial condition, we can obtain Sign Moment Theorem.

Proof of Absolute Value Moment Theorem

Lemma.

$$\lim_{s_1 \to t_1} \cdots \lim_{s_n \to t_n} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \mathbb{E}[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)]$$
$$= \left(\frac{2}{\pi}\right)^{n/2} \sqrt{\det \Sigma(\mathbf{t})}\, \mathbb{E}[|f(t_1) \cdots f(t_n)|]$$

Here we take the limit on the domain satisfying
$-1 < t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n < 1$.

Proof of Absolute Value Moment Theorem

Proof of Lemma (heuristic).

$$\frac{\partial^n}{\partial t_1 \cdots \partial t_n} \mathbb{E}[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)]$$

$$= 2^n \mathbb{E}[\delta_0(f(t_1)) f'(t_1) \cdots \delta_0(f(t_n)) f'(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)]$$

$$= 2^n \mathbb{E}[f'(t_1) \cdots f'(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n) \mid f(t_1) = \cdots = f(t_n) = 0] \\ \times (2\pi)^{-n/2} (\det \Sigma(\mathbf{t}))^{-1/2}$$

$$= \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{1/2} \cdot \mathbb{E}[f(t_1) \cdots f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)].$$

Taking the limits $s_i \to t_i$ gives the lemma since $|x| = (\operatorname{sgn} x)x$. $\square$

Proof of Absolute Value Moment Theorem

It follows from the previous lemma and Sign Moment Theorem that

$$\mathbb{E}[|f(t_1)\cdots f(t_n)|] =$$
$$\left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \lim_{\substack{s_i\to t_i\\ 1\le i\le n}} \frac{\partial^n}{\partial t_1\cdots\partial t_n} \operatorname{Pf}\left(\mathcal{K}(i,j)\right)_{1\le i,j\le n},$$

where

$$\mathcal{K}(i,j) = \begin{pmatrix} \mathbb{K}_{22}(t_i,t_j) & \mathbb{K}_{22}(t_i,s_j) \\ \mathbb{K}_{22}(s_i,t_j) & \mathbb{K}_{22}(s_i,s_j) \end{pmatrix}.$$

Thus

$$\lim_{\substack{s_i\to t_i\\ 1\le i\le n}} \frac{\partial^n}{\partial t_1\cdots\partial t_n} \operatorname{Pf}\left(\mathcal{K}(i,j)\right)_{1\le i,j\le n} = \operatorname{Pf}\left(\mathbb{K}(t_i,t_j)\right)_{1\le i,j\le n}.$$

Hence

$$\mathbb{E}[|f(t_1)\cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \operatorname{Pf}\left(\mathbb{K}(t_i,t_j)\right)_{1\le i,j\le n}.$$

Proof of Real Zero Correlation Theorem

Recall the correlation function $\rho_n^{\mathrm{r}}(t_1, \dots, t_n)$ for the real zeros of the random power series $f(z)$.

Lemma [Hammersley (1954)].

$$\rho_n^{\mathrm{r}}(t_1, \dots, t_n) = \frac{\mathbb{E}[|f'(t_1) \cdots f'(t_n)| \mid f(t_1) = \cdots = f(t_{2n}) = 0]}{(2\pi)^{n/2}\sqrt{\det \Sigma(\mathbf{t})}}$$

Using the lemma for the conditional expectation,

$$\rho_n^{\mathrm{r}}(t_1, \dots, t_n) = (2\pi)^{-n/2}(\det \Sigma(\mathbf{t}))^{1/2} \cdot \mathbb{E}[|f(t_1)f(t_2) \cdots f(t_n)|].$$

Hence it follows from Absolute Value Moment Theorem that

$$\rho_n^{\mathrm{r}}(t_1, \dots, t_n) = \pi^{-n} \operatorname{Pf}(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}.$$

Proof of Complex Zero Correlation Theorem

Recall the correlation function $\rho_n^{\mathrm{c}}(z_1, \ldots, z_n)$ for the complex zeros of the random power series $f(z)$.

Unlike $f(t)$ $(-1 < t < 1)$, the random variable $f(z)$ $(z \in \mathbb{D} \setminus \mathbb{R})$ is not Gaussian but $\Re f(z)$ and $\Im f(z)$ are real Gaussian.
Hammersley's formula gives

$$\rho_n^{\mathrm{c}}(z_1, \ldots, z_n) = \mathbb{E}[|f'(z_1) \cdots f'(z_n)|^2 \mid f(z_1) = \cdots = f(z_n) = 0] \cdot p_f(\mathbf{0}),$$

where $p_f(\mathbf{0})$ is the density of the Gaussian vector

$$(\Re f(z_1), \Im f(z_1), \ldots, \Re f(z_n), \Im f(z_n))$$

at $(0, 0, \ldots, 0, 0)$.

Proof of Complex Zero Correlation Theorem

Put

$$M = \left(\frac{1}{1 - z_i \bar{z}_j} \right)_{1 \le i,j \le 2n}, \qquad A = \left(\frac{1}{1 - z_i z_j} \right)_{1 \le i,j \le 2n}$$

with $z_{n+j} := \bar{z}_j$ $(j = 1, 2, \ldots, n)$.
It is easy to see that $p_f(\mathbf{0}) = \pi^{-n} (\det M)^{-1/2}$.
We can see that

$$\begin{aligned} &\mathbb{E}[|f'(z_1) \cdots f'(z_n)|^2 \mid f(z_1) = \cdots = f(z_n) = 0] \\ =&(-1)^n (\det A) \mathbb{E}[|f(z_1) \cdots f(z_n)|^2]. \end{aligned}$$

The mean value can be computed by Wick formula:

$$\mathbb{E}[|f(z_1) \cdots f(z_n)|^2] = \mathrm{Hf}\, A.$$

Proof of Complex Zero Correlation Theorem

Hence

$$\rho_n^{\mathrm{c}}(z_1, \dots, z_n) = \frac{(-1)^n (\det A)(\mathrm{Hf}\, A)}{\pi^n \sqrt{\det M}}.$$

Applying Ishikawa-Kawamuko-Okada formula, we obtain

$$\rho_n^{\mathrm{c}}(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^{n} \frac{1}{|1 - z_j^2|} \times (-1)^{n(n-1)/2}\, \mathrm{Pf} \left(\frac{z_i - z_j}{(1 - z_i z_j)^2} \right)_{1 \le i,j \le 2n}$$

with $z_{n+j} = \bar{z}_j$. Changing the order of rows/columns in the Pfaffian, we finish the proof of Complex Zero Correlation Theorem.

まとめ

次のランダムべき級数を考えた:

$$f(z) := \sum_{k=0}^{\infty} a_k z^k.$$

ここで，係数 $\{a_k\}_{k=0}^{\infty}$ は独立同分布で，実標準正規分布 $\mathrm{N}_{\mathbb{R}}(0,1)$ に従う．
収束半径は，ほとんど確実に 1 となる．

定理 (Complex Zero Correlation Theorem).

f の複素零点の相関関数は次で与えられる: $z_1, \dots, z_n$ を絶対値が 1 未満で，かつ虚部が正であるような複素数とするとき，

$$\rho_n^{\mathrm{c}}(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^{n} \frac{1}{|1 - z_j^2|} \cdot \mathrm{Pf}(\mathbb{K}^{\mathrm{c}}(z_i, z_j))_{1 \le i,j \le n}.$$

ここで，

$$\mathbb{K}^{\mathrm{c}}(z, w) = \begin{pmatrix} \frac{z-w}{(1-zw)^2} & \frac{z-\bar{w}}{(1-z\bar{w})^2} \\ \frac{\bar{z}-w}{(1-\bar{z}w)^2} & \frac{\bar{z}-\bar{w}}{(1-\bar{z}\bar{w})^2} \end{pmatrix}.$$

定理 (Real Zero Correlation Theorem).

f の実零点の相関関数は次で与えられる: $t_1, \ldots, t_n$ を区間 $(-1, +1)$ 内の実数とするとき,

$$\rho_n^{\mathrm{r}}(t_1, \ldots, t_n) = \pi^{-n} \operatorname{Pf}(\mathbb{K}(t_i, t_j))_{1 \le i,j \le n}.$$

ここで，各 $\mathbb{K}(s,t) = \begin{pmatrix} \mathbb{K}_{11}(s,t) & \mathbb{K}_{12}(s,t) \\ \mathbb{K}_{21}(s,t) & \mathbb{K}_{22}(s,t) \end{pmatrix}$ は 2×2 の行列で,

$$\mathbb{K}_{11}(s,t) = \frac{s-t}{\sqrt{(1-s^2)(1-t^2)}\,(1-st)^2},$$

$$\mathbb{K}_{12}(s,t) = \sqrt{\frac{1-t^2}{1-s^2}}\frac{1}{1-st},$$

$$\mathbb{K}_{21}(s,t) = -\sqrt{\frac{1-s^2}{1-t^2}}\frac{1}{1-st},$$

$$\mathbb{K}_{22}(s,t) = \operatorname{sgn}(t-s) \arcsin\left(\frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}\right).$$

定理 (Absolute Value Moment Theorem.)

$t_1, \dots, t_n$ を区間 $(-1, +1)$ 内の実数で互いに異なるとする．このとき，次が成り立つ．

$$\mathbb{E}[|f(t_1)\cdots f(t_n)|] = \left(\tfrac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2}\,\mathrm{Pf}(\mathbb{K}(t_i, t_j))_{1\le i,j\le n}.$$

ここで $\Sigma(\mathbf{t}) = \left(\frac{1}{1-t_i t_j}\right)_{1\le i,j\le n}$.

定理 (Sign Moment Theorem.)

$-1 < t_1 < \cdots < t_{2n} < 1$ ならば

$$\begin{aligned}\mathbb{E}[\mathrm{sgn}\, f(t_1)\,\mathrm{sgn}\, f(t_2)\cdots \mathrm{sgn}\, f(t_{2n})] &= \mathrm{Pf}\left(\mathbb{E}[\mathrm{sgn}\, f(t_i)\,\mathrm{sgn}\, f(t_j)]\right)_{1\le i<j\le 2n}\\ &= \left(\frac{2}{\pi}\right)^{n} \mathrm{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1\le i,j\le 2n}.\end{aligned}$$

終.

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2. Main results
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   - Proof of Real Zero Correlation Theorem
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4. Conclusion