

# Pfaffian formulas related to Gaussian power series

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# Gaussian power series

Consider the random power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Here  $\{a_k\}_{k=0}^{\infty}$  are i.i.d. (independent identically distributed)  $N_{\mathbb{R}}(0, 1)$ .

The radius of convergence is 1 almost surely.

In the restriction to real numbers, the process  $\{f(t)\}_{t \in (-1, +1)}$  is a mean zero real Gaussian process with covariance

$$\mathbb{E}[f(s)f(t)] = \sum_{k,l} s^k t^l \mathbb{E}[a_k a_l] = \sum_k (st)^k = \frac{1}{1-st}.$$

**Question.** How are the zeros of  $f$  distributed?

# Related works (1)

(1) Kac (1943). Consider a random polynomial

$$p_n(t) = \sum_{k=0}^n a_k t^k, \quad \text{where } \{a_k\}_{k=0}^n \text{ are i.i.d. } N_{\mathbb{R}}(0, 1).$$

Let  $N_n(\Omega)$  be the number of the **real** zeros of  $p_n$  in any measurable set  $\Omega$  in  $\mathbb{R}$ . Then

$$\mathbb{E}[N_n(\Omega)] = \frac{1}{\pi} \int_{\Omega} \sqrt{\frac{1}{(t^2 - 1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2} - 1)^2}} dt.$$

Note that  $\mathbb{E}[N_n(\mathbb{R})] \sim \frac{2}{\pi} \log n$  as  $n \rightarrow \infty$ .

The extension of this formula to **complex** zeros of  $p_n$  is obtained by Shepp-Vanderbei (1995).

## Related works (2)

(2) (Random Matrix Theory). Consider the characteristic polynomial

$$\phi_X(\lambda) = \det(\lambda I - X),$$

where  $X$  is an  $N \times N$  random matrix.

zeros of  $\phi_X \iff$  eigenvalues of  $X$ .

Suppose that  $X$  is a random matrix picked up from the Gaussian Orthogonal Ensemble (GOE), i.e.,

$$X = \frac{1}{2}(A + {}^tA), \quad A = (a_{ij}), \quad \{a_{ij}\} \text{ are i.i.d. } N_{\mathbb{R}}(0, 1).$$

Eigenvalues of  $X$  are real and the density is

$$C_N e^{-\sum \lambda_j^2/2} \prod_{i < j} |\lambda_i - \lambda_j|.$$

## Related works (2)

The distribution of the eigenvalues is described by correlation functions.

For each  $n = 1, 2, \dots, N$ , the  **$n$ -point correlation function** is defined as

$$\begin{aligned} & \rho_n(\lambda_1, \dots, \lambda_n) \\ &= \text{“The probability that there is an eigenvalue at each } \lambda_j, 1 \leq j \leq n\text{”} \\ &= \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} C_N e^{-\sum_{j=1}^N \lambda_j^2/2} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| d\lambda_{n+1} \cdots d\lambda_N. \end{aligned}$$

It is well known that  $\rho_n$  is given by a **Pfaffian**.

Thus, Zeros of the characteristic polynomial  $\phi_X$  (or eigenvalues of a GOE matrix  $X$ ) form a **Pfaffian point process**.

## Related works (3)

(3) Peres-Virág (2005). Consider a random power series

$$f_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \zeta_k z^k, \quad \{\zeta_k\}_{k=0}^{\infty}: \text{i.i.d. } N_{\mathbb{C}}(0, 1).$$

Consider the  $n$ -point correlation function for zeros of  $f_{\mathbb{C}}$ :

$$\rho_n(z_1, \dots, z_n)$$

= “The probability that  $f_{\mathbb{C}}$  has a zero at each  $z_j$ ,  $1 \leq j \leq n$ ”

for  $z_1, \dots, z_n$  in the open unit disc. Then

$$\rho_n(z_1, \dots, z_n) = \det \left( \frac{1}{(1 - z_i \bar{z}_j)^2} \right)_{1 \leq i, j \leq n}$$

Thus, the zeros of  $f_{\mathbb{C}}$  form a **determinantal point process**.

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# Pfaffian and Hafnian

Let

$$F_n = \left\{ \sigma \in S_{2n} \mid \begin{array}{l} \sigma(2i-1) < \sigma(2i) \ (i = 1, 2, \dots, n), \\ \sigma(1) < \sigma(3) < \dots < \sigma(2n-1) \end{array} \right\}$$

The **Pfaffian** of a skew-symmetric matrix  $B = (b_{ij})_{1 \leq i, j \leq 2n}$  is

$$\text{Pf } B = \sum_{\sigma \in F_n} (\text{sgn } \sigma) b_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)}.$$

The **Hafnian** of a symmetric matrix  $A = (a_{ij})_{1 \leq i, j \leq 2n}$  is

$$\text{Hf } A = \sum_{\sigma \in F_n} a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2n-1)\sigma(2n)}.$$

# Setting

We consider a real version of Peres-Virág (2005). Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \{a_k\}_{k=0}^{\infty}: \text{i.i.d. } N_{\mathbb{R}}(0, 1).$$

This random series converges on  $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$  almost surely.

We study the  **$n$ -point correlation functions** for **real** zeros and **complex** zeros

$$\rho_n^{\mathbb{r}}(t_1, \dots, t_n) \quad \text{and} \quad \rho_n^{\mathbb{c}}(z_1, \dots, z_n),$$

respectively. Here  $t_1, t_2, \dots, t_n$  are real numbers in  $\mathbb{D} \cap \mathbb{R}$ , and  $z_1, \dots, z_n$  are complex numbers in  $\mathbb{D}$  with  $\Im(z_i) > 0$ .

**Remark.** the correlation functions determine the distribution of zeros of  $f$ .

# Real zero correlation

## Theorem (Real Zero Correlation Theorem).

The  $n$ -point correlation function for the real zeros of  $f$  is given by

$$\rho_n^r(t_1, \dots, t_n) = \pi^{-n} \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

Here each  $\mathbb{K}(s, t) = \begin{pmatrix} \mathbb{K}_{11}(s, t) & \mathbb{K}_{12}(s, t) \\ \mathbb{K}_{21}(s, t) & \mathbb{K}_{22}(s, t) \end{pmatrix}$  is a  $2 \times 2$  matrix defined by

$$\mathbb{K}_{11}(s, t) = \frac{s - t}{\sqrt{(1 - s^2)(1 - t^2)} (1 - st)^2}, \quad \mathbb{K}_{12}(s, t) = \sqrt{\frac{1 - t^2}{1 - s^2}} \frac{1}{1 - st},$$

$$\mathbb{K}_{21}(s, t) = -\sqrt{\frac{1 - s^2}{1 - t^2}} \frac{1}{1 - st},$$

$$\mathbb{K}_{22}(s, t) = \text{sgn}(t - s) \arcsin \left( \frac{\sqrt{(1 - s^2)(1 - t^2)}}{1 - st} \right). \quad (\text{Put } \text{sgn } 0 = 0.)$$

# Complex zero correlation

## Theorem (Complex Zero Correlation Theorem).

The  $n$ -point correlation function for the complex zeros of  $f$  is given by

$$\rho_n^c(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^n \frac{1}{|1 - z_j^2|} \cdot \text{Pf}(\mathbb{K}^c(z_i, z_j))_{1 \leq i, j \leq n}.$$

Here each  $\mathbb{K}^c(z, w)$  is a  $2 \times 2$  matrix defined by

$$\mathbb{K}^c(z, w) = \begin{pmatrix} \frac{z-w}{(1-zw)^2} & \frac{z-\bar{w}}{(1-z\bar{w})^2} \\ \frac{\bar{z}-w}{(1-\bar{z}w)^2} & \frac{\bar{z}-\bar{w}}{(1-\bar{z}\bar{w})^2} \end{pmatrix}$$

**Example (one-point correlations).**  $\rho_1^r(t) = \frac{1}{\pi(1-t^2)}$  and

$$\rho_1^c(z) = \frac{1}{\pi\sqrt{-1}} \frac{1}{|1 - z^2|} \frac{z - \bar{z}}{(1 - z\bar{z})^2} = \frac{|z - \bar{z}|}{\pi|1 - z^2|(1 - |z|^2)^2}.$$

# Remarks on Real/Complex Zero Correlations

- The theorems say that the distribution of zeros of  $f$  form a Pfaffian point process.
- The kernel  $\mathbb{K}(s, t)$  satisfies  $\mathbb{K}_{ab}(s, t) = -\mathbb{K}_{ba}(t, s)$  and

$$\mathbb{K}_{11}(s, t) = \frac{\partial^2}{\partial s \partial t} \mathbb{K}_{22}(s, t) \quad \mathbb{K}_{12}(s, t) = \frac{\partial}{\partial s} \mathbb{K}_{22}(s, t)$$

$$\mathbb{K}_{21}(s, t) = \frac{\partial}{\partial t} \mathbb{K}_{22}(s, t) \quad \mathbb{K}_{22}(s, t) =$$

$$\mathbb{K}_{22}(s, t) = \operatorname{sgn}(t - s) \arcsin \frac{\sqrt{(1 - s^2)(1 - t^2)}}{1 - st}.$$

# Remarks on Real/Complex Zero Correlations

The last two theorems are not new results. Forrester (2010 arXiv) obtained them independently. His proof is quite different from ours. He proved them by using the following facts.

- The distribution of zeros of  $f$  coincides with the limit eigenvalue distribution of a [truncated Haar orthogonal matrix](#) [Krishnapur (2009)].
- The eigenvalue distribution can be studied in the standard way developed by Forrester, Nagao, Borodin, etc. The correlations for eigenvalues are given in terms of Pfaffians in explicit ways.

In our proof, random matrix theory is not used. One of the key in our proof is a Hafnian-Pfaffian formula due to Ishikawa-Kawamuko-Okada (2005).

# Absolute Value Moment Theorem

In the course of our proof of Real Zero Correlation Theorem, we will obtain the following theorems.

Let  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  be as before. For distinct real numbers  $-1 < t_1, \dots, t_n < 1$ , denote by  $\Sigma(\mathbf{t})$  the covariance matrix for the mean zero Gaussian vector  $(f(t_1), \dots, f(t_n))$ :

$$\Sigma(\mathbf{t}) = (\mathbb{E}[f(t_i)f(t_j)])_{1 \leq i, j \leq n} = \left( \frac{1}{1 - t_i t_j} \right)_{1 \leq i, j \leq n}.$$

## Theorem (Absolute Value Moment Theorem).

$$\mathbb{E}[|f(t_1) \cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

Here  $\mathbb{K}(s, t)$  is given in Real Zero Correlation Theorem.

# Sign Moment Theorem

## Theorem (Sign Moment Theorem).

If  $-1 < t_1 < \cdots < t_{2n} < 1$ , then

$$\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1 \leq i, j \leq 2n},$$

where  $\mathbb{K}_{22}(s, t) = \operatorname{sgn}(t - s) \arcsin \frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}$ . Hence

$$\mathbb{E}[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = \operatorname{Pf}(\mathbb{E}[\operatorname{sgn} f(t_i) \operatorname{sgn} f(t_j)])_{1 \leq i < j \leq 2n}.$$

Note:  $\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_n)] = 0$  if  $n$  is an odd number.

# Example

**Example.** For  $-1 < t_1 < t_2 < t_3 < t_4 < 1$ ,

$$\begin{aligned} & \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \operatorname{sgn} f(t_3) \operatorname{sgn} f(t_4)] \\ &= \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2)] \cdot \mathbb{E}[\operatorname{sgn} f(t_3) \operatorname{sgn} f(t_4)] \\ &\quad - \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_3)] \cdot \mathbb{E}[\operatorname{sgn} f(t_2) \operatorname{sgn} f(t_4)] \\ &\quad + \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_4)] \cdot \mathbb{E}[\operatorname{sgn} f(t_2) \operatorname{sgn} f(t_3)] \end{aligned}$$

and

$$\mathbb{E}[\operatorname{sgn} f(t_i) \operatorname{sgn} f(t_j)] = \frac{2}{\pi} \arcsin \frac{\sqrt{(1-t_i^2)(1-t_j^2)}}{1-t_it_j}.$$

# Wick formula

Recall the well-known Wick formula.

## Wick formula.

For a mean zero real Gaussian vector  $(X_1, X_2, \dots, X_{2n})$ , we have

$$\mathbb{E}[X_1 X_2 \cdots X_{2n}] = \text{Hf}(\mathbb{E}[X_i X_j])_{1 \leq i, j \leq 2n}$$

and  $\mathbb{E}[X_1 \cdots X_m] = 0$  if  $m$  is odd.

For example,

$$\begin{aligned} \mathbb{E}[X_1 X_2 X_3 X_4] &= \mathbb{E}[X_1 X_2] \cdot \mathbb{E}[X_3 X_4] \\ &\quad + \mathbb{E}[X_1 X_3] \cdot \mathbb{E}[X_2 X_4] + \mathbb{E}[X_1 X_4] \cdot \mathbb{E}[X_2 X_3]. \end{aligned}$$

# Remarks

Let  $(X_1, X_2, \dots, X_n)$  be a mean zero real Gaussian vector with covariance  $\sigma_{ij} = \mathbb{E}[X_i X_j]$ .

Are there any Wick-type formula for absolute value moment  $\mathbb{E}[|X_1 \cdots X_n|]$  or sign moment  $\mathbb{E}[\text{sgn } X_1 \cdots \text{sgn } X_n]$ ?

No.

**Proposition [Nabeya (1951)].**

$$\mathbb{E}[|X_1 X_2|] = \frac{2}{\pi} \left( \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2} + \sigma_{12} \arcsin \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \right)$$

where  $\sigma_{ij} = \mathbb{E}[X_i X_j]$ .

Nabeya (1952) obtained a similar but more complicated formula for the  $n = 3$  case. Any  $n \geq 4$  formula is not known and can not be expected.

## Known facts

If  $n$  is odd, then  $\mathbb{E}[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_n] = 0$ .

$$\mathbb{E}[\operatorname{sgn} X_1 \operatorname{sgn} X_2] = \frac{2}{\pi} \arcsin \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}},$$

where  $\sigma_{ij} = \mathbb{E}[X_i X_j]$ .

There is no such formula  $\mathbb{E}[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_{2n}]$  with  $2n \geq 4$ .

Thus, our theorems give Pfaffian expressions for the absolute value moment and sign moment in the special case

$$\mathbb{E}[X_i X_j] = \frac{1}{1 - t_i t_j}.$$

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# Differential Sign Moment Formula

We first prove the formula for the derivation of the sign moment.  
Recall

$$\mathbb{K}_{11}(s, t) = \frac{s - t}{\sqrt{(1 - s^2)(1 - t^2)}(1 - st)^2}.$$

Proposition (Differential Sign Moment Formula).

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}_{11}(t_i, t_j))_{1 \leq i, j \leq 2n} \\ &= \left(\frac{2}{\pi}\right)^n \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - t_i^2}} \cdot \operatorname{Pf} \left( \frac{t_i - t_j}{(1 - t_i t_j)^2} \right)_{1 \leq i, j \leq 2n}. \end{aligned}$$

# Proof of Differential Sign Moment Formula

Since  $\frac{d}{dx} \operatorname{sgn} x = 2\delta_0(x)$ , where  $\delta_0$  is the Dirac delta function, we have

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= 2^{2n} \mathbb{E}[\delta_0(f(t_1)) f'(t_1) \cdots \delta_0(f(t_{2n})) f'(t_{2n})] \\ &= \frac{2^{2n} \mathbb{E}[f'(t_1) \cdots f'(t_{2n}) \mid f(t_1) = \cdots = f(t_{2n}) = 0]}{(2\pi)^n (\det \Sigma(\mathbf{t}))^{1/2}}, \end{aligned}$$

where  $\Sigma(\mathbf{t}) = ((1 - t_i t_j)^{-1})_{1 \leq i, j \leq 2n}$ .

Note that  $(2\pi)^{-n} (\det \Sigma(\mathbf{t}))^{-1/2}$  is the density of the Gaussian vector  $(f(t_1), \dots, f(t_{2n}))$  at  $(0, \dots, 0)$ .

This heuristic discussion can be justified in the framework of Malliavin calculus.

# Proof of Differential Sign Moment Formula

## Lemma.

The conditional distribution of  $(f'(t_1), \dots, f'(t_{2n}))$  given  $f(t_1) = \dots = f(t_{2n}) = 0$  has the same distribution with

$$(q_1(\mathbf{t})f(t_1), \dots, q_{2n}(\mathbf{t})f(t_{2n})).$$

Here

$$q_i(\mathbf{t}) = \frac{1}{1 - t_i^2} \prod_{\substack{1 \leq k \leq 2n \\ k \neq i}} \frac{t_i - t_k}{1 - t_i t_k}.$$

Note:

$$(-1)^n q_1(\mathbf{t}) \cdots q_{2n}(\mathbf{t}) = \frac{\prod_{1 \leq i < j \leq 2n} (t_i - t_j)^2}{\prod_{i,j=1}^{2n} (1 - t_i t_j)} = \det \left( \frac{1}{1 - t_i t_j} \right)_{1 \leq i, j \leq 2n}.$$

# Proof of Differential Sign Moment Formula

Hence,

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= (-1)^n \left(\frac{2}{\pi}\right)^n (\det \Sigma(\mathbf{t}))^{1/2} \cdot \mathbb{E}[f(t_1) \cdots f(t_{2n})]. \end{aligned}$$

Cauchy's determinant formula and Wick formula give

$$= \left(\frac{2}{\pi}\right)^n \prod_{i=1}^{2n} \frac{1}{\sqrt{1-t_i^2}} \cdot \prod_{1 \leq i < j \leq 2n} \frac{t_i - t_j}{1 - t_i t_j} \cdot \operatorname{Hf} \left( \frac{1}{1 - t_i t_j} \right)_{1 \leq i, j \leq 2n}.$$

Recall Schur's pfaffian

$$\operatorname{Pf} \left( \frac{t_i - t_j}{1 - t_i t_j} \right)_{1 \leq i, j \leq 2n} = \prod_{1 \leq i < j \leq 2n} \frac{t_i - t_j}{1 - t_i t_j}.$$

# Proof of Differential Sign Moment Formula

Lemma [Ishikawa-Kawamuko-Okada (2005)].

$$\text{Pf} \left( \frac{t_i - t_j}{1 - t_i t_j} \right)_{i,j} \cdot \text{Hf} \left( \frac{1}{1 - t_i t_j} \right)_{i,j} = \text{Pf} \left( \frac{t_i - t_j}{(1 - t_i t_j)^2} \right)_{i,j}.$$

Hence we have

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \mathbb{E}[\text{sgn } f(t_1) \text{sgn } f(t_2) \cdots \text{sgn } f(t_{2n})] \\ &= \left(\frac{2}{\pi}\right)^n \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - t_i^2}} \cdot \text{Pf} \left( \frac{t_i - t_j}{(1 - t_i t_j)^2} \right)_{1 \leq i, j \leq 2n}. \end{aligned}$$

We thus finish the proof of the Differential Sign Moment Formula.

# Proof of Sign Moment Theorem

## Sign Moment Theorem.

For  $-1 < t_1 < \cdots < t_{2n-1} < t_{2n} < 1$ ,

$$\mathbb{E}[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1 \leq i, j \leq 2n},$$

where  $\mathbb{K}_{22}(s, t) = \operatorname{sgn}(t - s) \arcsin \frac{\sqrt{(1-s^2)(1-t^2)}}{1-st}$ .

**Proof.** Check

$$\frac{\partial^2}{\partial s \partial t} \mathbb{K}_{22}(s, t) = \mathbb{K}_{11}(s, t) = \frac{s - t}{\sqrt{(1-s^2)(1-t^2)}(1-st)^2}.$$

If we integrate Differential Sign Moment Formula with a good initial condition, we can obtain Sign Moment Theorem.

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# Proof of Absolute Value Moment Theorem

## Lemma.

$$\begin{aligned} & \lim_{s_1 \rightarrow t_1} \cdots \lim_{s_n \rightarrow t_n} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \mathbb{E}[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)] \\ &= \left(\frac{2}{\pi}\right)^{n/2} \sqrt{\det \Sigma(\mathbf{t})} \mathbb{E}[|f(t_1) \cdots f(t_n)|] \end{aligned}$$

Here we take the limit on the domain satisfying  
 $-1 < t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n < 1$ .

# Proof of Absolute Value Moment Theorem

Proof of Lemma (heuristic).

$$\begin{aligned} & \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \mathbb{E}[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)] \\ &= 2^n \mathbb{E}[\delta_0(f(t_1)) f'(t_1) \cdots \delta_0(f(t_n)) f'(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)] \\ &= 2^n \mathbb{E}[f'(t_1) \cdots f'(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n) \mid f(t_1) = \cdots = f(t_n) = 0] \\ & \quad \times (2\pi)^{-n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \\ &= \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{1/2} \cdot \mathbb{E}[f(t_1) \cdots f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)]. \end{aligned}$$

Taking the limits  $s_i \rightarrow t_i$  gives the lemma since  $|x| = (\operatorname{sgn} x)x$ .  $\square$

# Proof of Absolute Value Moment Theorem

It follows from the previous lemma and Sign Moment Theorem that

$$\mathbb{E}[|f(t_1) \cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \lim_{\substack{s_i \rightarrow t_i \\ 1 \leq i \leq n}} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \text{Pf}(\mathcal{K}(i, j))_{1 \leq i, j \leq n},$$

where

$$\mathcal{K}(i, j) = \begin{pmatrix} \mathbb{K}_{22}(t_i, t_j) & \mathbb{K}_{22}(t_i, s_j) \\ \mathbb{K}_{22}(s_i, t_j) & \mathbb{K}_{22}(s_i, s_j) \end{pmatrix}.$$

Thus

$$\lim_{\substack{s_i \rightarrow t_i \\ 1 \leq i \leq n}} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \text{Pf}(\mathcal{K}(i, j))_{1 \leq i, j \leq n} = \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

Hence

$$\mathbb{E}[|f(t_1) \cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

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# Proof of Real Zero Correlation Theorem

Recall the correlation function  $\rho_n^r(t_1, \dots, t_n)$  for the real zeros of the random power series  $f(z)$ .

Lemma [Hammersley (1954)].

$$\rho_n^r(t_1, \dots, t_n) = \frac{\mathbb{E}[|f'(t_1) \cdots f'(t_n)| \mid f(t_1) = \cdots = f(t_n) = 0]}{(2\pi)^{n/2} \sqrt{\det \Sigma(\mathbf{t})}}$$

Using the lemma for the conditional expectation,

$$\rho_n^r(t_1, \dots, t_n) = (2\pi)^{-n/2} (\det \Sigma(\mathbf{t}))^{1/2} \cdot \mathbb{E}[|f(t_1)f(t_2) \cdots f(t_n)|].$$

Hence it follows from Absolute Value Moment Theorem that

$$\rho_n^r(t_1, \dots, t_n) = \pi^{-n} \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

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# Proof of Complex Zero Correlation Theorem

Recall the correlation function  $\rho_n^c(z_1, \dots, z_n)$  for the complex zeros of the random power series  $f(z)$ .

Unlike  $f(t)$  ( $-1 < t < 1$ ), the random variable  $f(z)$  ( $z \in \mathbb{D} \setminus \mathbb{R}$ ) is **not** Gaussian but  $\Re f(z)$  and  $\Im f(z)$  are **real Gaussian**.  
**Hammersley's formula** gives

$$\rho_n^c(z_1, \dots, z_n) = \mathbb{E}[|f'(z_1) \cdots f'(z_n)|^2 \mid f(z_1) = \cdots = f(z_n) = 0] \cdot p_f(\mathbf{0}),$$

where  $p_f(\mathbf{0})$  is the density of the Gaussian vector

$$(\Re f(z_1), \Im f(z_1), \dots, \Re f(z_n), \Im f(z_n))$$

at  $(0, 0, \dots, 0, 0)$ .

# Proof of Complex Zero Correlation Theorem

Put

$$M = \left( \frac{1}{1 - z_i \bar{z}_j} \right)_{1 \leq i, j \leq 2n}, \quad A = \left( \frac{1}{1 - z_i z_j} \right)_{1 \leq i, j \leq 2n}$$

with  $z_{n+j} := \bar{z}_j$  ( $j = 1, 2, \dots, n$ ).

It is easy to see that  $p_f(\mathbf{0}) = \pi^{-n}(\det M)^{-1/2}$ .

We can see that

$$\begin{aligned} & \mathbb{E}[|f'(z_1) \cdots f'(z_n)|^2 \mid f(z_1) = \cdots = f(z_n) = 0] \\ &= (-1)^n (\det A) \mathbb{E}[|f(z_1) \cdots f(z_n)|^2]. \end{aligned}$$

The mean value can be computed by Wick formula:

$$\mathbb{E}[|f(z_1) \cdots f(z_n)|^2] = \text{Hf } A.$$

# Proof of Complex Zero Correlation Theorem

Hence

$$\rho_n^c(z_1, \dots, z_n) = \frac{(-1)^n (\det A) (\text{Hf } A)}{\pi^n \sqrt{\det M}}.$$

Applying Ishikawa-Kawamuko-Okada formula, we obtain

$$\begin{aligned} \rho_n^c(z_1, \dots, z_n) &= \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^n \frac{1}{|1 - z_j^2|} \\ &\quad \times (-1)^{n(n-1)/2} \text{Pf} \left( \frac{z_i - z_j}{(1 - z_i z_j)^2} \right)_{1 \leq i, j \leq 2n} \end{aligned}$$

with  $z_{n+j} = \bar{z}_j$ . Changing the order of rows/columns in the Pfaffian, we finish the proof of Complex Zero Correlation Theorem.

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We have studied the random power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Here  $\{a_k\}_{k=0}^{\infty}$  are i.i.d.  $N_{\mathbb{R}}(0, 1)$ .

The radius of convergence of  $f$  is 1 almost surely.

## Complex Zero Correlation Theorem.

Let  $z_1, \dots, z_n$  be complex numbers and assume  $|z_i| < 1$  and  $\Im(z_i) > 0$  for all  $i$ . The  $n$ -point correlation function for the complex zeros of  $f$  is given by

$$\rho_n^c(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^n \frac{1}{|1 - z_j^2|} \cdot \text{Pf}(\mathbb{K}^c(z_i, z_j))_{1 \leq i, j \leq n}$$

with

$$\mathbb{K}^c(z, w) = \begin{pmatrix} \frac{z-w}{(1-zw)^2} & \frac{z-\bar{w}}{(1-z\bar{w})^2} \\ \frac{\bar{z}-w}{(1-\bar{z}w)^2} & \frac{\bar{z}-\bar{w}}{(1-\bar{z}\bar{w})^2} \end{pmatrix}.$$

## Real Zero Correlation Theorem.

Let  $t_1, \dots, t_n$  be real numbers satisfying  $-1 < t_i < 1$  for all  $i$ . The  $n$ -point correlation function for the real zeros of  $f$  is given by

$$\rho_n^r(t_1, \dots, t_n) = \pi^{-n} \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

Here each  $\mathbb{K}(s, t) = \begin{pmatrix} \mathbb{K}_{11}(s, t) & \mathbb{K}_{12}(s, t) \\ \mathbb{K}_{21}(s, t) & \mathbb{K}_{22}(s, t) \end{pmatrix}$  is a  $2 \times 2$  matrix defined by

$$\mathbb{K}_{11}(s, t) = \frac{s - t}{\sqrt{(1 - s^2)(1 - t^2)}(1 - st)^2}, \quad \mathbb{K}_{12}(s, t) = \sqrt{\frac{1 - t^2}{1 - s^2}} \frac{1}{1 - st},$$

$$\mathbb{K}_{21}(s, t) = -\sqrt{\frac{1 - s^2}{1 - t^2}} \frac{1}{1 - st},$$

$$\mathbb{K}_{22}(s, t) = \text{sgn}(t - s) \arcsin \frac{\sqrt{(1 - s^2)(1 - t^2)}}{1 - st}.$$

# Example

$$\begin{aligned}\rho_2^r(t_1, t_2) &= \frac{1}{\pi^2} \text{Pf} \begin{pmatrix} 0 & \mathbb{K}_{12}(t_1, t_1) & \mathbb{K}_{11}(t_1, t_2) & \mathbb{K}_{12}(t_1, t_2) \\ & 0 & \mathbb{K}_{21}(t_1, t_2) & \mathbb{K}_{22}(t_1, t_2) \\ & & 0 & \mathbb{K}_{12}(t_2, t_2) \\ & & & 0 \end{pmatrix} \\ &= \frac{1}{\pi^2} (\mathbb{K}_{12}(t_1, t_1)\mathbb{K}_{12}(t_2, t_2) - \mathbb{K}_{11}(t_1, t_2)\mathbb{K}_{22}(t_1, t_2) \\ &\quad + \mathbb{K}_{12}(t_1, t_2)\mathbb{K}_{21}(t_1, t_2)) \\ &= \frac{(t_1 - t_2)^2}{\pi^2(1 - t_1^2)(1 - t_2^2)(1 - t_1 t_2)^2} \\ &\quad + \frac{|t_1 - t_2|}{\pi^2 \sqrt{(1 - t_1^2)(1 - t_2^2)}(1 - t_1 t_2)^2} \arcsin \frac{\sqrt{(1 - t_1^2)(1 - t_2^2)}}{1 - t_1 t_2}.\end{aligned}$$

## Absolute Value Moment Theorem.

Suppose  $t_1, \dots, t_n$  are distinct. Put  $\Sigma(\mathbf{t}) = \left( \frac{1}{1-t_i t_j} \right)_{1 \leq i, j \leq n}$ .

$$\mathbb{E}[|f(t_1) \cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-1/2} \text{Pf}(\mathbb{K}(t_i, t_j))_{1 \leq i, j \leq n}.$$

## Sign Moment Theorem.

If  $-1 < t_1 < \cdots < t_{2n} < 1$ , then

$$\begin{aligned} \mathbb{E}[\text{sgn } f(t_1) \text{sgn } f(t_2) \cdots \text{sgn } f(t_{2n})] &= \text{Pf}(\mathbb{E}[\text{sgn } f(t_i) \text{sgn } f(t_j)])_{1 \leq i < j \leq 2n} \\ &= \left(\frac{2}{\pi}\right)^n \text{Pf}(\mathbb{K}_{22}(t_i, t_j))_{1 \leq i, j \leq 2n}. \end{aligned}$$

# Thank you for listening.

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