Moments of matrix elements from the orthogonal group and COE

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Random matrix

Consider a random matrix

$$X = (x_{ij})_{1 \le i,j \le N}.$$

(ex. Gaussian matrix, Wishart matrix, Haar-distributed unitary matrix, etc.)

Question

How can we compute the following mixed moments ?

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n}]$$

or

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} \overline{x_{k_1 l_1} x_{k_2 l_2} \cdots x_{k_n l_n}}]$$

History

- **Gaussian matrix**: well-known Wick formula.
  If Z_1, Z_2, Z_3, Z_4 are Gaussian r.v., then

$$\mathbb{E}[Z_1 Z_2 Z_3 Z_4] = \mathbb{E}[Z_1 Z_2]\mathbb{E}[Z_3 Z_4] + \mathbb{E}[Z_1 Z_3]\mathbb{E}[Z_2 Z_4] + \mathbb{E}[Z_1 Z_4]\mathbb{E}[Z_2 Z_3].$$

- **central complex Wishart matrix and its inverse matrix**: [Graczyk-Letac-Massam 03].
- **central real Wishart matrix and its inverse matrix**: [Graczyk-Letac-Massam 05], [M 11].
- **noncentral Wishart matrix**: [Kuriki-Numata 10].

History

- **Haar-distributed unitary matrix**: [Samuel 80], [Weingarten 78], [Collins 03]. We call their technique Weingarten calculus.
- **Haar-distributed orthogonal matrix**: [Collins-Śniady 06], [Collins-M 09] — Today's topic 1.
- **Dyson's circular ensembles**:
  - circular unitary ensemble (CUE) = Unitary group with Haar measure.
  - circular orthogonal ensemble (COE) — Today's topic 2.
  - circular symplectic ensemble (CSE) — in future.

Haar-distributed orthogonal matrix

$$O(N) = \{N \times N \text{ real orthogonal matrices}\}$$

There exists a unique probability measure dR on $O(N)$ such that

$$\int_{O(N)} f(R_1 R R_2) dR = \int_{O(N)} f(R) dR$$

for all $R_1, R_2 \in O(N)$ and integrable functions f.
We call dR the Haar measure for $O(N)$.

Consider the probability space $(\, O(N), \mathrm{Borel}, \, dR \,)$. The coordinate functions

$$r_{ij} := \text{ the } (i,j)\text{-entry of a Haar-distributed random matrix } R$$

are random variables.

Our first question

Let $R = (r_{ij})_{1 \le i,j \le N}$ be a Haar-distributed orthogonal matrix.

Question 1

How do we compute the following moments ?

$$\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_k j_k}] = \int_{O(N)} r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_k j_k} dR$$

where $i_1, \dots, i_k, j_1, \dots, j_k \in [N] := \{1, 2, \dots, N\}$.

Note that $\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_k j_k}]$ vanishes if k is odd.

Perfect matchings

Let $\mathcal{M}(2k)$ be the set of all perfect matchings

$$\mathfrak{m} = \{\{\mathfrak{m}(1), \mathfrak{m}(2)\}, \{\mathfrak{m}(3), \mathfrak{m}(4)\}, \dots, \{\mathfrak{m}(2k-1), \mathfrak{m}(2k)\}\}$$

on $[2k] = \{1, 2, \dots, 2k\}$. Here we write as
$\mathfrak{m}(2i-1) < \mathfrak{m}(2i)$ $(1 \le i \le k)$ and $\mathfrak{m}(1) < \mathfrak{m}(3) < \cdots < \mathfrak{m}(2k-1)$.

Example. $\mathcal{M}(4)$ consists of three elements

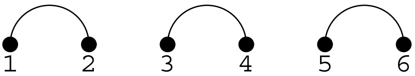
$$\{\{1,2\},\{3,4\}\}, \quad \{\{1,3\},\{2,4\}\}, \quad \{\{1,4\},\{2,3\}\}.$$

Each $\mathfrak{m} \in \mathcal{M}(2k)$ can be identified with a permutation

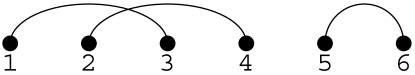
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2k-1 & 2k \\ \mathfrak{m}(1) & \mathfrak{m}(2) & \mathfrak{m}(3) & \mathfrak{m}(4) & \cdots & \mathfrak{m}(2k-1) & \mathfrak{m}(2k) \end{pmatrix} \in S_{2k}.$$

Thus $\mathcal{M}(2k) \subset S_{2k}$.

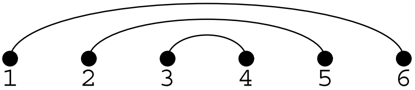
$$\{\{1,2\},\{3,4\},\{5,6\}\} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$



$$\{\{1,3\},\{2,4\},\{5,6\}\} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix}$$



$$\{\{1,6\},\{2,5\},\{3,4\}\} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 5 & 3 & 4 \end{pmatrix}$$



Collins-Śniady theorem

Theorem [Collins-Śniady (2006)]

Let $R = (r_{ij})_{1 \le i,j \le N}$ be a Haar-distributed orthogonal matrix. For sequences $\mathbf{i} = (i_1, \dots, i_{2k})$, $\mathbf{j} = (j_1, \dots, j_{2k})$ in $[N]^{\times 2k}$,

$$\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_{2k} j_{2k}}] = \sum_{\mathfrak{m}, \mathfrak{n} \in \mathcal{M}(2n)} \delta_{\mathfrak{m}}(\mathbf{i}) \delta_{\mathfrak{n}}(\mathbf{j}) \mathrm{Wg}_k^{O(N)}(\mathfrak{m}^{-1}\mathfrak{n}).$$

Here

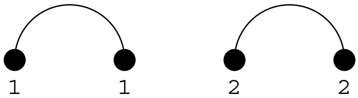
$$\delta_{\mathfrak{m}}(\mathbf{i}) := \begin{cases} 1 & \text{if } i_p = i_q \text{ for all matchings } \{p, q\} \text{ in } \mathfrak{m}, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{Wg}_k^{O(N)}$ is a function on S_{2k} called the orthogonal Weingarten function. We will give the definition later.

Example 1

Let $R = (r_{ij})_{1 \le i,j \le N}$ be a Haar-distributed orthogonal matrix and consider $\mathbb{E}[r_{11}^2 r_{22}^2] = \mathbb{E}[r_{11} r_{11} r_{22} r_{22}]$.

$(i_1, \dots, i_4) = (1, 1, 2, 2)$. $(j_1, \dots, j_4) = (1, 1, 2, 2)$.
If $\delta_{\mathfrak{m}}((1, 1, 2, 2)) = 1$, then $\mathfrak{m} \in \mathcal{M}(4)$ must be



$$\mathbb{E}[r_{11}^2 r_{22}^2] = \mathrm{Wg}_2^{O(N)}(\mathfrak{m}^{-1}\mathfrak{m}) = \mathrm{Wg}_2^{O(N)}(\mathrm{id}_4)$$
$$= \frac{N+1}{N(N-1)(N+2)}.$$

Example 2

Consider $\mathbb{E}[r_{11} r_{12} r_{21} r_{22}]$.

$\mathbf{i} = (1, 1, 2, 2)$ and $\mathbf{j} = (1, 2, 1, 2)$.
If $\delta_{\mathfrak{m}}(\mathbf{i}) = 1$ and $\delta_{\mathfrak{n}}(\mathbf{j}) = 1$, then $\mathfrak{m}$ and $\mathfrak{n}$ must be

,

respectively.

Hence

$$\mathbb{E}[r_{11} r_{12} r_{21} r_{22}] = \mathrm{Wg}_2^{O(N)}(\mathfrak{m}^{-1}\mathfrak{n}) = \frac{-1}{N(N-1)(N+2)}.$$

Example 3, 4

Consider $\mathbb{E}[r_{22}^4]$.
$\mathbf{i} = \mathbf{j} = (2,2,2,2)$, and so $\delta_{\mathfrak{m}}(\mathbf{i}) = 1$ for any $\mathfrak{m} \in \mathcal{M}(4)$.
Hence

$$\mathbb{E}[r_{22}^4] = \sum_{\mathfrak{m},\mathfrak{n} \in \mathcal{M}(4)} \mathrm{Wg}_2^{O(N)}(\mathfrak{m}^{-1}\mathfrak{n}) = \frac{3}{N(N+2)}.$$

Consider $\mathbb{E}[r_{11}^3 r_{12}]$.

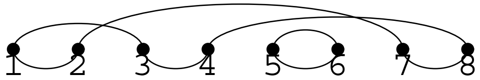
$\mathbf{i} = (1,1,1,1)$, $\mathbf{j} = (1,1,1,2)$.
There exist no $\mathfrak{n} \in \mathcal{M}(4)$ such that $\delta_{\mathfrak{n}}(\mathbf{j}) = 1$.
Hence

$$\mathbb{E}[r_{11}^3 r_{12}] = 0.$$

Graph associated with permutations

For each permutation $\sigma \in S_{2k}$, we consider the graph $\Gamma(\sigma)$ defined as follows:
Vertex set $\{1, 2, \ldots, 2k\}$;
Edges $\{\sigma(2i-1), \sigma(2i)\}, \{2i-1, 2i\} \quad i = 1, 2, \ldots, k$.
Let $\kappa(\sigma)$ be the number of connected components in $\Gamma(\sigma)$.

Example. Let $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 6 & 5 & 2 & 7 & 4 & 8 \end{smallmatrix}\right) \in S_8$. Then $\kappa(\sigma) = 2$.



First definition

Let $G_k^{O(N)}$ be the function on S_{2k} defined by

$$G_k^{O(N)}(\sigma) = N^{\kappa(\sigma)}.$$

Consider the convolution product for functions on S_{2k}:

$$(f * g)(\sigma) = \sum_{\tau \in S_{2k}} f(\sigma\tau^{-1})g(\tau) \quad (\sigma \in S_{2k}).$$

Definition (due to [Collins–Śniady (2006)])

The orthogonal Weingarten function $\mathrm{Wg}_k^{O(N)}$ is, by definition, the (pseudo-)inverse element of $G_k^{O(N)}$ with respect to the convolution product.

Remark. We will give more explicit expression for $\mathrm{Wg}_k^{O(N)}$ later.

Hyperoctahedral group

The hyperoctahedral group H_k is the subgroup of S_{2k} generated by

$$\begin{aligned} (2i-1\ 2i), &\qquad 1 \le i \le k, \\ (2i-1\ 2j-1)(2i\ 2j), &\qquad 1 \le i < j \le k. \end{aligned}$$

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of k is a sequence of positive integers such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0, \quad k = \sum_{i=1}^{l} \lambda_i.$$

Some symbols

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of k. Define

$$C'_\lambda(N) := \prod_{i=1}^{l} \prod_{j=1}^{\lambda_i} (N + 2j - i - 1).$$

The zonal spherical function for the Gelfand pair (S_{2k}, H_k) is defined by

$$\omega^\lambda(\sigma) := (2^k k!)^{-1} \sum_{\zeta \in H_k} \chi^{2\lambda}(\sigma\zeta), \qquad (\sigma \in S_{2k})$$

where $\chi^{2\lambda}$ is the irreducible character of S_{2k} associated with $2\lambda = (2\lambda_1, 2\lambda_2, \dots, 2\lambda_l)$.

$$\begin{aligned} f^{2\lambda} :=& \chi^{2\lambda}(\mathrm{id}_{S_{2k}}) \\ =& \#\text{ of standard Young tableaux of shape } 2\lambda. \end{aligned}$$

Second definition

Definition & Theorem (due to [Collins-M (2009)])

Define

$$\mathrm{Wg}_k^{O(N)}(\sigma) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{C'_\lambda(N)} \omega^\lambda(\sigma) \qquad (\sigma \in S_{2k})$$

summed over all partitions λ of k. This definition coincides with the First definition.

Example ($k=2$). If $\sigma = (2\ 3) \in S_4$,

$$\mathrm{Wg}_2^{O(N)}(\sigma) = \frac{1}{3}\Big(\underbrace{\frac{1 \cdot 1}{N(N+2)}}_{\lambda=(2)} + \underbrace{\frac{2 \cdot (-\frac{1}{2})}{N(N-1)}}_{\lambda=(1,1)} \Big) = \frac{-1}{N(N+2)(N-1)}.$$

Circular Orthogonal Ensemble

$$\mathrm{COE}(N) = \{N \times N \text{ symmetric unitary matrices}\}$$

There exists a unique probability measure dV on $\mathrm{COE}(N)$ such that

$$\int_{\mathrm{COE}(N)} f(W^{\mathrm{T}} V W) dV = \int_{\mathrm{COE}(N)} f(V) dV$$

for all $N \times N$ unitary matrix W and integrable functions f.

Consider the probability space $(\,\mathrm{COE}(N), \mathrm{Borel}, dV\,)$. We call it the Circular Orthogonal Ensemble (COE) and V a COE matrix.

Circular ensembles

There are three classes of circular ensembles:
COE (circular orthogonal ensemble): symmetric unitary matrices
CUE (circular unitary ensemble): unitary matrices
CSE (circular symplectic ensemble): self-dual unitary quaternion matrices

It is well known that eigenvalue density function for a COE/CUE/CSE matrix is given by

$$C_{N\beta} \prod_{1 \le i < j \le N} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^{\beta}$$

with $\beta = 1, 2, 4$ respectively. The constant $C_{N\beta}$ is fixed by normalization.

Our second question

Let $V = (v_{ij})_{1 \le i,j \le N}$ be a COE matrix.

Question 2

How do we compute the following moments ?

$$\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2k-1} i_{2k}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2k-1} j_{2k}}}]$$

where $i_1, \ldots, i_{2k}, j_1, \ldots, j_{2k} \in [N]$.

Note that $\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2k-1} i_{2k}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2l-1} j_{2l}}}]$ vanishes unless $k = l$.

Theorem

Theorem [M (2011 preprint)]

Let $V = (v_{ij})_{1 \le i,j \le N}$ be a COE matrix. For sequences $\mathbf{i} = (i_1, \dots, i_{2k})$, $\mathbf{j} = (j_1, \dots, j_{2k})$ in $[N]^{\times 2k}$,

$$\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2k-1} i_{2k}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2k-1} j_{2k}}}] = \sum_{\substack{\sigma \in S_{2k} \\ \mathbf{j} = \mathbf{i}^\sigma}} \mathrm{Wg}_k^{O(N+1)}(\sigma),$$

where the sum runs over permutations $\sigma \in S_{2k}$ satisfying

$$\mathbf{j} = \mathbf{i}^\sigma := (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(2k)}),$$

and $\mathrm{Wg}_k^{O(N+1)}$ is the orthogonal Weingarten function with parameter $N+1$ (not N).

Example

Example.

$$\mathbb{E}[v_{12} v_{34} \overline{v_{13} v_{24}}] \qquad (\mathbf{i} = (1,2,3,4),\ \mathbf{j} = (1,3,2,4))$$
$$= \mathrm{Wg}_2^{O(N+1)} \left(\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \right) = \frac{-1}{N(N+1)(N+3)}.$$

Example.

$$\mathbb{E}[|v_{11}|^4] = \mathbb{E}[v_{11} v_{11} \overline{v_{11} v_{11}}] \qquad (\mathbf{i} = \mathbf{j} = (1,1,1,1))$$
$$= \sum_{\sigma \in S_4} \mathrm{Wg}_2^{O(N+1)}(\sigma) = \frac{8}{(N+1)(N+3)}.$$

Example.

$$\mathbb{E}[v_{12}^2 \overline{v_{11} v_{12}}] = 0$$

because $\mathbf{j} = (1,1,1,2)$ is not a rearrangement of $\mathbf{i} = (1,2,1,2)$.

Example

Example.

$$\mathbb{E}[v_{12}^2\,\overline{v_{11}\,v_{22}}] \qquad (\mathbf{i}=(1,2,1,2),\ \mathbf{j}=(1,1,2,2))$$
$$=\sum_{\sigma\in L}\mathrm{Wg}_2^{O(N+1)}(\sigma)=\frac{-4}{N(N+1)(N+3)},$$

where $L=\{\left(\begin{smallmatrix}1&2&3&4\\1&3&2&4\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1&2&3&4\\3&1&2&4\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1&2&3&4\\1&3&4&2\end{smallmatrix}\right),\ \left(\begin{smallmatrix}1&2&3&4\\3&1&4&2\end{smallmatrix}\right)\}$.

Example.

$$\mathbb{E}[v_{12}\,v_{34}\,v_{56}\,v_{78}\,\overline{v_{18}\,v_{23}\,v_{45}\,v_{67}}]$$
$$=\mathrm{Wg}_4^{O(N+1)}\left(\left(\begin{smallmatrix}1&2&3&4&5&6&7&8\\1&8&2&3&4&5&6&7\end{smallmatrix}\right)\right)$$
$$=\frac{-5N-11}{(N-2)(N-1)N(N+1)(N+2)(N+3)(N+5)(N+7)}.$$

Corollary 1 (known fact)

Let v_{ii} be a diagonal entry of an $N \times N$ COE matrix. Then, for $k \geq 1$,

$$\mathbb{E}[|v_{ii}|^{2k}] = \frac{2^k k!}{(N+1)(N+3)\cdots(N+2k-1)}.$$

Proof.

$$\begin{aligned}
&\mathbb{E}[|v_{ii}|^{2k}] \\
=&\sum_{\sigma \in S_{2k}} \mathrm{Wg}_k^{O(N+1)}(\sigma) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{C'_\lambda(N+1)} \sum_{\sigma \in S_{2k}} \omega^\lambda(\sigma) \\
=&2^k k! \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{C'_\lambda(N+1)} \delta_{\lambda,(k)} = \frac{2^k k!}{(N+1)(N+3)\cdots(N+2k-1)}.
\end{aligned}$$

Applications

Corollary 2 (new result)

Let v_{ij} be an off-diagonal entry of an $N \times N$ COE matrix. Then, for $k \geq 1$,

$$\mathbb{E}[|v_{ij}|^{2k}] = \frac{k!}{N(N+1)(N+2)\cdots(N+k-2)\cdot(N+2k-1)}.$$

Proof. $\mathbf{i} = \mathbf{j} = (i, j, i, j, \ldots, i, j)$.

$$\mathbb{E}[|v_{ij}|^{2k}] = \sum_{\sigma} \mathrm{Wg}_k^{O(N+1)}(\sigma),$$

summed over all $\sigma \in S_{2k}$, each of which permutates odd numbers and even numbers. The computation in this case is more complicated than Corollary 1. But we can obtain the claim.

Asymptotics

Let $R^{(N)}$, $N = 1, 2, \ldots$, be a sequence of Haar-distributed orthogonal matrices of size N. We would like to know the asymptotics

$$\mathbb{E}[r_{i_1 j_1}^{(N)} \cdots r_{i_{2k}, j_{2k}}^{(N)}], \qquad N \to \infty$$

where $i_1, \ldots, i_{2k}, j_1, j_2, \ldots, j_{2k}$ are fixed.

For example, we have already known

$$\mathbb{E}[(r_{11}^{(N)})^4] = \frac{3}{N(N+2)} = \frac{3}{N^2} - \frac{6}{N^3} + \frac{12}{N^4} + O(N^{-5}).$$

Asymptotics

To know the asymptotic behavior of orthogonal matrix integrals, we need to know the asymptotic behavior of the orthogonal Weingarten function.
However, the expression (second definition)

$$\mathrm{Wg}_k^{O(N)}(\sigma) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{C'_\lambda(N)} \omega^\lambda(\sigma) \qquad (\sigma \in S_{2k})$$

is not suitable for our purpose. We need the third expression for $\mathrm{Wg}_k^{O(N)}$.

Asymptotics

Theorem [M (2011)]

Fix $\sigma \in S_{2k}$. Expand as

$$\mathrm{Wg}_k^{O(N)}(\sigma) = \sum_{m=0}^{\infty} (-1)^m a_m(\sigma) N^{-k-m}.$$

Then $a_m(\sigma)$ has the following combinatorial interpretation (see the next slide). In particular,

- all $a_m(\sigma)$ are non-negative integers.
- $a_m(\sigma) = 0$ unless $m \geq k - \kappa(\sigma)$.
- If $m = k - \kappa(\sigma)$, then $a_m(\sigma) = \prod_{j=1}^{\kappa(\sigma)} \mathrm{Cat}_{\mu_j - 1}$. Here $\mathrm{Cat}_r = \frac{(2r)!}{(r+1)!\, r!}$ is the Catalan number, and $\mu = (\mu_1, \mu_2, \dots)$ is a partition of k, determined by σ.

Combinatorial interpretation

Let $k, m \geq 1$ and $\sigma \in S_{2k}$. The coefficient $a_m(\sigma)$ is the number of sequences $(s_1, t_1, s_2, t_2, \ldots, s_m, t_m)$ of positive integers satisfying

- t_i are odd, and $1 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq 2k-1$.
- $s_i < t_i$.
- Let $\rho := (s_1\ t_1) \cdots (s_m\ t_m)$ be the permutation in S_{2k} defined as the product of transpositions $(s_i\ t_i)$. Then two associated perfect matchings

$$\{\{\rho(1), \rho(2)\}, \{\rho(3), \rho(4)\}, \ldots, \{\rho(2k-1), \rho(2k)\}\},$$
$$\{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \ldots, \{\sigma(2k-1), \sigma(2k)\}\}$$

coincide.

Example

Recall that

$$\mathbb{E}[r_{11}^2 r_{22}^2 \cdots r_{kk}^2] = \mathrm{Wg}_k^{O(N)}(\mathrm{id}_{2k}).$$
$$\mathbb{E}[|v_{12} v_{34} \cdots v_{2k-1,2k}|^2] = \mathrm{Wg}_k^{O(N+1)}(\mathrm{id}_{2k}).$$

Here $R = (r_{ij})$ is an $N \times N$ Haar-distributed orthogonal matrix and $V = (v_{ij})$ is an $N \times N$ COE matrix.

$$\mathrm{Wg}_k^{O(N)}(\mathrm{id}_{2k}) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{C'_\lambda(N)}$$
$$= N^{-k} + k(k-1)N^{-k-2} - k(k-1)N^{-k-3} + O(N^{-k-4})$$

as $N \to \infty$.

Summary

We have established the systematic method of the computations for moments of two random matrices.

Let $\mathbf{i} = (i_1, \dots, i_{2k})$, $\mathbf{j} = (j_1, \dots, j_{2k})$ be two sequences in $[N]^{\times 2k}$.

For a Haar-distributed orthogonal matrix $R = (r_{ij})_{1 \le i,j \le N}$,

$$\mathbb{E}[r_{i_1 j_1} r_{i_2 j_2} \cdots r_{i_{2k} j_{2k}}] = \sum_{\mathfrak{m},\mathfrak{n} \in \mathcal{M}(2n)} \delta_{\mathfrak{m}}(\mathbf{i}) \delta_{\mathfrak{n}}(\mathbf{j}) \mathrm{Wg}_k^{O(N)}(\mathfrak{m}^{-1}\mathfrak{n}).$$

For a COE matrix $V = (v_{ij})_{1 \le i,j \le N}$,

$$\mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2k-1} i_{2k}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2k-1} j_{2k}}}] = \sum_{\substack{\sigma \in S_{2k} \\ \mathbf{j} = \mathbf{i}^{\sigma}}} \mathrm{Wg}_k^{O(N+1)}(\sigma).$$

Summary

The orthogonal Weingarten function $\mathrm{Wg}_k^{O(N)}$ have three forms.

1. It is the pseudo-inverse element of $G_k^{O(N)}(\sigma) = N^{\kappa(\sigma)}$.

2. $$\mathrm{Wg}_k^{O(N)}(\sigma) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} \frac{f^{2\lambda}}{C'_\lambda(N)} \omega^\lambda(\sigma).$$

3. $$\mathrm{Wg}_k^{O(N)}(\sigma) = \sum_{m=0}^{\infty} (-1)^m a_m(\sigma) N^{-k-m}$$

where the coefficients $a_m(\sigma)$ have combinatorial interpretations.