On Moments of entries of a COE matrix

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Random matrix

Consider a random matrix

$$X = (x_{ij})_{1 \le i,j \le N}.$$

(ex. Gaussian matrix, Wishart matrix, Haar-distributed unitary matrix, etc.)

Problem

How can we compute the following mixed moments ?

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n}]$$

or

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} \overline{x_{k_1 l_1} x_{k_2 l_2} \cdots x_{k_n l_n}}]$$

History

- **Gaussian matrix**: well-known Wick formula.
  If y_1, y_2, y_3, y_4 are Gaussian r.v., then

$$\begin{aligned}\mathbb{E}[y_1 y_2 y_3 y_4] = \mathbb{E}[y_1 y_2]\mathbb{E}[y_3 y_4] \\ + \mathbb{E}[y_1 y_3]\mathbb{E}[y_2 y_4] + \mathbb{E}[y_1 y_4]\mathbb{E}[y_2 y_3].\end{aligned}$$

- **central complex Wishart matrix and its inverse matrix**: [Graczyk-Letac-Massam 03].
- **central real Wishart matrix and its inverse matrix**: [Graczyk-Letac-Massam 05], [M 11].
- **noncentral Wishart matrix**: [Kuriki-Numata 10].

History

- **Haar-distributed unitary matrix**: [Samuel 80], [Weingarten 78], [Collins 03]. We call their technique Weingarten calculus.
- **Haar-distributed orthogonal matrix**: [Collins-Śniady 06], [Collins-M 09].
- **Dyson's circular ensembles**:
  - circular unitary ensemble (CUE) = Unitary group with Haar measure.
  - circular orthogonal ensemble (COE) — Today's topic.
  - circular symplectic ensemble (CSE) — in future.

Definition of COE

$$\mathrm{COE}(N) := \{N \times N \text{ symmetric unitary matrices }\}.$$

Fact and Definition

There is a unique random matrix $V \in \mathrm{COE}(N)$ such that

$$V \text{ and } W_0^{\mathrm{T}} V W_0 \text{ have the same distribution}$$

where W_0 is an $N \times N$ fixed unitary matrix.
We call this V a COE matrix.

Recall that a CUE matrix (or a Haar-distributed unitary matrix) U is a random matrix in the unitary group $U(N)$ such that

$$U \text{ and } W_1 U W_2 \text{ have the same distribution}$$

where W_1, W_2 are $N \times N$ fixed unitary matrices.

Our Problem

Let $V = (v_{ij})_{1 \le i,j \le N}$ be a COE matrix. Let $\mathbf{i} = (i_1, i_2, \dots, i_{2n})$ and $\mathbf{j} = (j_1, j_2, \dots, j_{2n})$ be two sequences in $\{1, 2, \dots, N\}^{\times 2n}$.
Let

$$M_N(\mathbf{i}, \mathbf{j}) := \mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2n-1} i_{2n}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2n-1} j_{2n}}}].$$

Example.

$$M_N((1234), (1324)) = \mathbb{E}[v_{12} v_{34} \overline{v_{13} v_{24}}].$$
$$M_N((1212), (1212)) = \mathbb{E}[v_{12} v_{12} \overline{v_{12} v_{12}}] = \mathbb{E}[|v_{12}|^4].$$

Problem

Give a method for computing the moments $M_N(\mathbf{i}, \mathbf{j})$.

Main Theorem

Main Theorem

Let $M_N(\mathbf{i},\mathbf{j})$ be as above. Then we have

$$M_N(\mathbf{i},\mathbf{j}) = \sum_{\substack{\sigma \in S_{2n} \\ \mathbf{j}=\mathbf{i}^\sigma}} \mathrm{Wg}_n^{O(N+1)}(\sigma),$$

where the sum runs over permutations σ in the symmetric group S_{2n} satisfying

$$\mathbf{j} = \mathbf{i}^\sigma := (i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(2n)}),$$

and $\mathrm{Wg}_n^{O(N+1)}$ is the orthogonal Weingarten function. (We will give the definition of the orthogonal Weingarten function later.)

How to use Main Theorem

Example.

$$\mathbb{E}[v_{12} v_{34} \overline{v_{13} v_{24}}] = M_N((1234),(1324))$$
$$= \mathrm{Wg}_2^{O(N+1)}\left(\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}\right) = \frac{-1}{N(N+1)(N+3)}.$$

Example.

$$\mathbb{E}[|v_{11}|^2] = \mathbb{E}[v_{11} v_{11} \overline{v_{11} v_{11}}] = M_N((1111),(1111))$$
$$= \sum_{\sigma \in S_4} \mathrm{Wg}_2^{O(N+1)}(\sigma) = \frac{8}{(N+1)(N+3)}.$$

Example. Since $\mathbf{j} = (1112)$ is not a rearrangement of $\mathbf{i} = (1212)$,

$$\mathbb{E}[v_{12}^2 \overline{v_{11} v_{12}}] = M_N((1212),(1112)) = 0.$$

Definition of Orthogonal Wg

The hyperoctahedral group H_n is the subgroup of S_{2n} generated by

$$\begin{array}{rl} (2k-1\ 2k), & 1 \le k \le n, \\ (2i-1\ 2j-1)(2i\ 2j) & 1 \le i < j \le n. \end{array}$$

Well-known Fact

Double cosets

$$\{H_n \sigma H_n \mid \sigma \in S_{2n}\}$$

are parametrized by partitions of n. Hence, we have the decomposition

$$S_{2n} = \bigsqcup_{\mu \vdash n} H_\mu,$$

where each H_μ is of the form $H_n \sigma H_n$ for some $\sigma \in S_{2n}$.

Definition of Orthogonal Wg

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n.

$$C'_\lambda(N) := \prod_{(i,j)\in\lambda} (N + 2j - i - 1),$$

where the product runs over all boxes of the Young diagram of λ.
The zonal spherical function for the pair (S_{2n}, H_n) is defined by

$$\omega^\lambda(\sigma) := (2^n n!)^{-1} \sum_{\tau\in H_n} \chi^{2\lambda}(\sigma\tau), \qquad (\sigma \in S_{2n})$$

where $\chi^{2\lambda}$ is the irreducible character of S_{2n} associated with $2\lambda = (2\lambda_1, 2\lambda_2, \dots) \vdash 2n$.

$$\begin{aligned} f^{2\lambda} :=& \chi^{2\lambda}(\mathrm{id}_{S_{2n}}) \\ =& \#\text{ of standard Young tableaux of shape } 2\lambda. \end{aligned}$$

Definition of Orthogonal Wg

Definition

The orthogonal Weingarten function is defined by

$$\mathrm{Wg}_n^{O(N)}(\sigma) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C'_\lambda(N)} \omega^\lambda(\sigma) \qquad (\sigma \in S_{2n}).$$

Example ($n = 2$). If $\sigma = (2\ 3) \in S_4$,

$$\mathrm{Wg}_2^{O(N)}(\sigma) = \frac{1}{3}\Bigg(\underbrace{\frac{1 \cdot 1}{N(N+2)}}_{\lambda=(2)} + \underbrace{\frac{2 \cdot (-\frac{1}{2})}{N(N-1)}}_{\lambda=(1,1)} \Bigg) = \frac{-1}{N(N+2)(N-1)}.$$

Example

Fact

$\mathrm{Wg}_n^{O(N)}$ takes constant at each double coset H_μ.

If $n = 3$, then $S_6 = H_{(3)} \sqcup H_{(2,1)} \sqcup H_{(1,1,1)}$.
$\mathrm{Wg}_3^{O(N)}(\sigma)$ are given by

$$\begin{aligned} &\frac{2}{N(N+2)(N+4)(N-1)(N-2)} && (\sigma \in H_{(3)}), \\ &\frac{-1}{N(N+4)(N-1)(N-2)} && (\sigma \in H_{(2,1)}), \\ &\frac{N^2+3N-2}{N(N+2)(N+4)(N-1)(N-2)} && (\sigma \in H_{(1,1,1)}). \end{aligned}$$

Main Theorem (again)

Main Theorem

Let $V = (v_{ij})_{1 \le i,j \le N}$ be a COE matrix. Let $\mathbf{i} = (i_1, i_2, \dots, i_{2n})$ and $\mathbf{j} = (j_1, j_2, \dots, j_{2n})$ be two sequences in $\{1, 2, \dots, N\}^{\times 2n}$, and let

$$M_N(\mathbf{i}, \mathbf{j}) := \mathbb{E}[v_{i_1 i_2} v_{i_3 i_4} \cdots v_{i_{2n-1} i_{2n}} \overline{v_{j_1 j_2} v_{j_3 j_4} \cdots v_{j_{2n-1} j_{2n}}}].$$

Then we have

$$M_N(\mathbf{i}, \mathbf{j}) = \sum_{\substack{\sigma \in S_{2n} \\ \mathbf{j} = \mathbf{i}^\sigma}} \mathrm{Wg}_n^{O(N+1)}(\sigma).$$

Haar-distributed orthogonal matrix

Compare Main Theorem with the following theorem.

Theorem. [Collins-Śniady (06), Collins-M (09)]

Let $X = (x_{ij})_{1 \le i,j \le N}$ be a Haar-distributed orthogonal matrix taken from $O(N)$. For two sequences $\mathbf{i} = (i_1, \dots, i_{2n})$ and $\mathbf{j} = (j_1, \dots, j_{2n})$,

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_{2n} j_{2n}}] = (2^n n!)^{-2} \sum_{\sigma} \sum_{\tau} \mathrm{Wg}_n^{O(N)}(\sigma^{-1}\tau),$$

where the sum runs over all $\sigma, \tau \in S_{2n}$ satisfying

$$i_{\sigma(2k-1)} = i_{\sigma(2k)}, \quad j_{\tau(2k-1)} = j_{\tau(2k)} \quad \text{for all } 1 \le k \le n.$$

Applications

Corollary 1

Let v_{ii} be a diagonal entry of an $N \times N$ COE matrix. Then, for $n \geq 1$,

$$\mathbb{E}[|v_{ii}|^{2n}] = \frac{2^n n!}{(N+1)(N+3)\cdots(N+2n-1)}.$$

Proof.

$$\mathbb{E}[|v_{ii}|^{2n}] = M_N((\underbrace{i,\ldots,i}_{2n}),(\underbrace{i,\ldots,i}_{2n}))$$

$$= \sum_{\sigma \in S_{2n}} \mathrm{Wg}_n^{O(N+1)}(\sigma) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C'_\lambda(N+1)} \sum_{\sigma \in S_{2n}} \omega^\lambda(\sigma)$$

$$= 2^n n! \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{C'_\lambda(N+1)} \delta_{\lambda,(n)} = \frac{2^n n!}{(N+1)(N+3)\cdots(N+2n-1)}.$$

Applications

Corollary 2

Let v_{ij} be an off-diagonal entry of an $N \times N$ COE matrix. Then, for $n \geq 1$,

$$\mathbb{E}[|v_{ij}|^{2n}] = \frac{n!}{N(N+1)(N+2)\cdots(N+n-2)\cdot(N+2n-1)}.$$

Proof.

$$\mathbb{E}[|v_{ij}|^{2n}] = M_N((i,j,i,j,\dots,i,j),(i,j,i,j,\dots,i,j))$$
$$= \sum_{\sigma} \mathrm{Wg}_n^{O(N+1)}(\sigma),$$

summed over all $\sigma \in S_{2n}$, each of which permutates odd numbers and even numbers. This case is more difficult than Corollary 1...

Conclusion

We have studied the following problem.

Problem

Given a random matrix $X = (x_{ij})_{1 \le i,j \le N}$, how can we compute the following mixed moments ?

$$\mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n}] \quad \text{or} \quad \mathbb{E}[x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} \overline{x_{k_1 l_1} x_{k_2 l_2} \cdots x_{k_n l_n}}]$$

This problem has been completely solved for:
Gaussian matrix, Wishart matrix, unitary group $U(N)$, orthogonal group $O(N)$...
Dyson's circular orthogonal ensemble $\mathrm{COE}(N)$ New!

The orthogonal Weingarten function $\mathrm{Wg}_n^{O(N+1)}$ is a key item.