Toric duality for deformations of the cyclic quotient singularities $A_{n,q}$ and $A_{n,n-q}$

Oswald Riemenschneider

Mathematisches Seminar
Universität Hamburg

Tuesday, October 12, 2010 / Lecture Kagoshima University

Motto

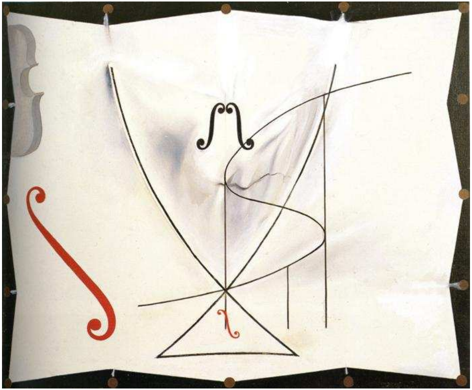
I like a good story well told. That is the reason I am sometimes forced to tell them myself.

Mark Twain

Motto

I like a good story well told. That is the reason I am sometimes forced to tell them myself.

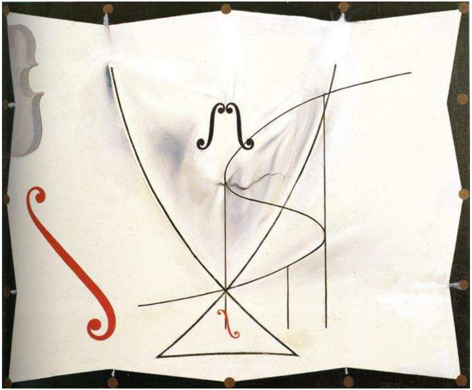
Mark Twain



Motto

I like a good story well told. That is the reason I am sometimes forced to tell them myself.

Mark Twain



Salvador Dali: *The swallow's tail*

Introduction

Pairs of *relatively prime* positive integers

$$(n,\ q)$$

with

$$q \ < \ n$$

classify simple *geometric*, *algebro - geometric*, *combinatoric*, *group theoretic* and *invariant theoretic* objects.
These are in several ways connected with each other by concrete constructions. Here, the *continued fraction expansions* of the rational numbers

$$\frac{n}{q} \qquad \text{resp.} \qquad \frac{n}{n \ - \ q}$$

play an essential role. Replacing q by $n - q$ reflects various types of *duality*.

Plan of lecture

1. Continued fraction expansions
2. Generation of certain semi - groups
3. Toric varieties of type $A_{n,q}$
4. Cyclic quotient surface singularities
5. Deformations
6. Duality for the Artin deformation

1. Continued fraction expansions

Hirzebruch - Jung expansion

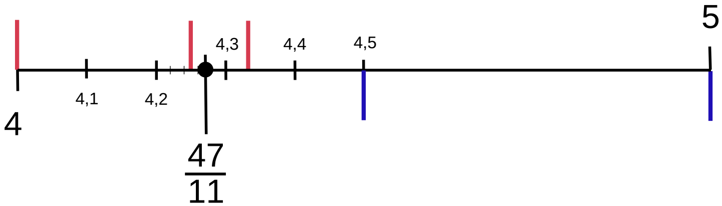
A variant of the Euclidean Algorithm is given by *Division "with smallest excess"*, i.e.

$$\begin{array}{rcrcrcl} n &=& b_1 q &-& q_1 \,, & 0 < q_1 < q & , \\ q &=& b_2 q_1 &-& q_2 \,, & 0 \le q_2 < q_1 & , \\ \dots & & \dots & & & \dots & \\ q_{r-2} &=& b_r q_{r-1} &-& q_r \,, & 0 = q_r & . \end{array}$$

which leads to a semi - regular continued fraction expansion:

$$n/q = b_1 - \underline{1 \,\rceil\, b_2} - \cdots - \underline{1 \,\rceil\, b_r} \,, \quad b_\rho \geq 2 \,.$$

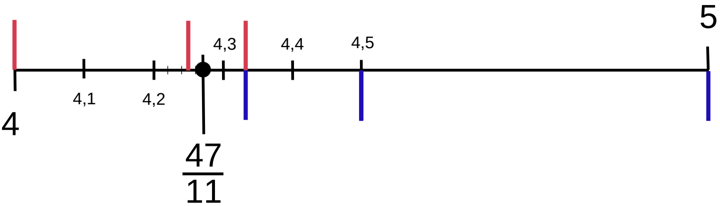
Example: 47/11



$5 \quad 5 - 1 | 2 = 9/2$

$47/11 = 5 - 1 | 2 - 1 | 2 - 1 | 3 - 1 | 2$

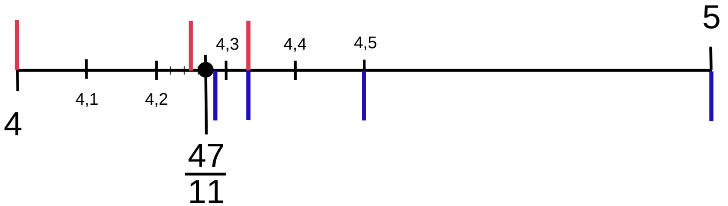
47/11 once more



5 $\quad 5 - 1 \,|\, 2 = 9/2 \qquad 5 - 1 \,|\, 2 - 1 \,|\, 2 = 13/3$

$47/11 = 5 - 1 \,|\, 2 - 1 \,|\, 2 - 1 \,|\, 3 - 1 \,|\, 2$

47/11 once more



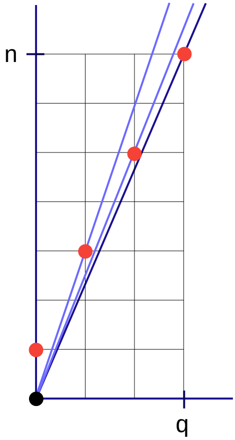
$$5 - \frac{1|}{|2} - \frac{1|}{|2} - \frac{1|}{|3} = 30/7$$

$$5 \qquad 5 - \frac{1|}{|2} = 9/2 \qquad 5 - \frac{1|}{|2} - \frac{1|}{|2} = 13/3$$

$$47/11 = 5 - \frac{1|}{|2} - \frac{1|}{|2} - \frac{1|}{|3} - \frac{1|}{|2}$$

2. Generation of certain semi - groups

Generation of certain semi - groups in $\mathbb{Z} \oplus \mathbb{Z}$

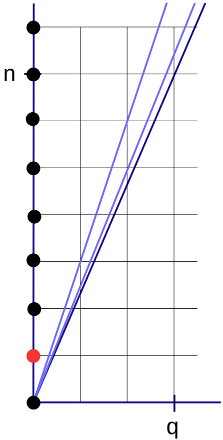


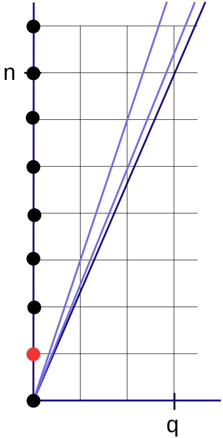
$7/3 = 3 - \cfrac{1}{2 - \cfrac{1}{2}}$

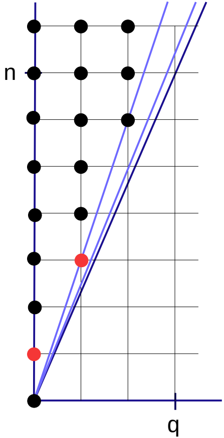
3

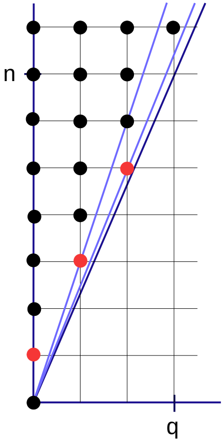
5/2

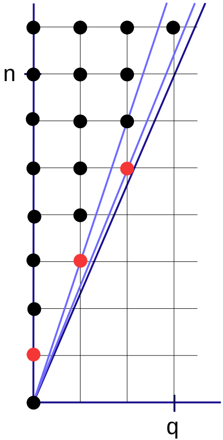
7/3

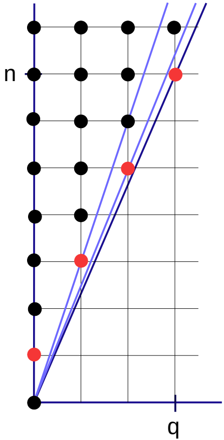

n
q


n
q


n
q


n
q


n
q


n
q

The generating lattice points Q_ρ, $\rho = 1, \ldots, r$, whose coordinates are exactly the nominator and denominator of the approximating fractions of n/q, are most simply obtained by the inductive definition

$$Q_{-1} := \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad Q_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Q_\rho := b_\rho \, Q_{\rho-1} - Q_{\rho-2} .$$

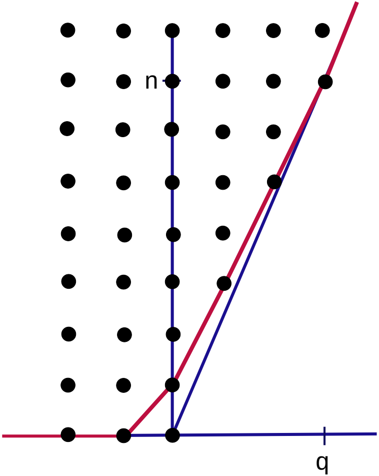
In our example we have the sequence 3, 2, 2 and - indeed - the generators

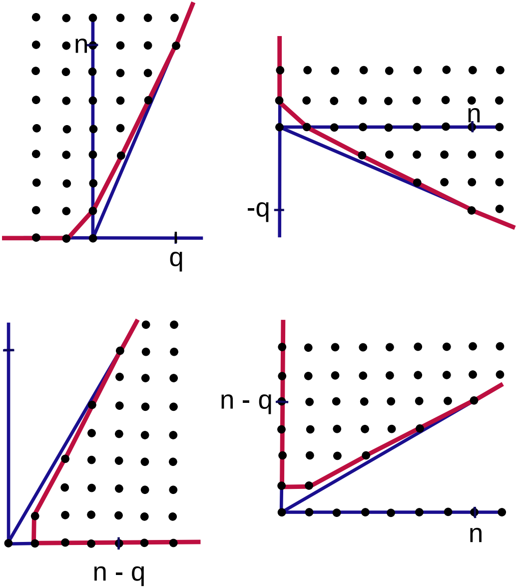
$$Q_{-1} := \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \, Q_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \, Q_1 := \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \, Q_2 := \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \, Q_3 := \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

Hence, it is more natural to regard the larger cone

$$\sigma := \{ (x, y) \in \mathbb{R}^2 : y \geq 0 , \; nx \leq qy \}$$

and the preceding sequence of generators for the semi - group $\sigma \cap (\mathbb{Z} \oplus \mathbb{Z})$.


n
q


n
q
n
-q
n
n - q
n - q
n

Duality

We are, however, mainly interested in the cones

$$\sigma_{n,q} := \{ (x, y) \in \mathbb{R}^2 : 0 \leq qy \leq nx \}$$

whose *dual* $\sigma_{n,q}^\vee$ of linear forms which are non negative on $\sigma_{n,q}$ is obviously given (after identifying $\mathbb{R}^2$ with its dual space):

$$\sigma_{n,q}^\vee := \{ (x, y) \in \mathbb{R}^2 : ny \geq -qx , \ x \geq 0 \} .$$

Corollary

$$\sigma_{n,q}^\vee \cong \sigma_{n,q}^c ,$$

where the superscript "c" denotes the complement in the upper half plane.

Corollary

The semi - group

$$\sigma_{n,q} \cap (\mathbb{Z} \oplus \mathbb{Z})$$

is minimally generated by the lattice points $P_\mu\,,\ \mu = 0, \ldots, m+1\,,$ *where*

$$P_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad P_{\mu+1} := a_\mu P_\mu - P_{\mu-1}$$

and

$$n/(n-q) = a_1 - \underline{1\ \rfloor}\, \overline{a_2} - \cdots - \underline{1\ \rfloor}\, \overline{a_m}\ .$$

3. Toric varieties of type $A_{n,q}$

Affine toric Varieties

Affine toric varieties of dimension N are combinatorically determined by so called *strongly convex rational polyhedral cones* σ in $\mathbb{R}^N$. In case $N = 2$, these are simply (closed semi -) cones that are bordered by two half - lines with rational slope and do not contain any line through the origin. In other words: There exist lattice points γ_1 and γ_2 in $\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ such that

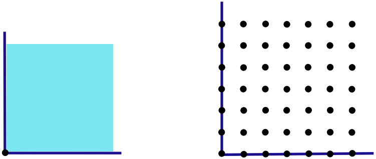
$$\sigma = \mathbb{R}_{\geq 0}\gamma_1 + \mathbb{R}_{\geq 0}\gamma_2 \quad \text{and} \quad \sigma \cap (-\sigma) = \{0\} .$$

If identifying the dual space of $\mathbb{R}^N$ with $\mathbb{R}^N$ itself, then the *dual cone* $\sigma^\vee$ of linear forms which are non negative on σ is of the same type, and the algebra

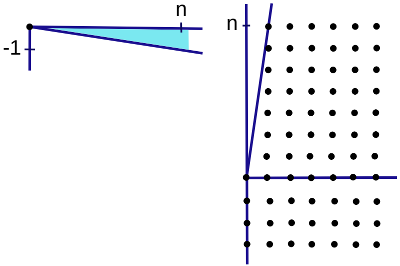
$$\mathbb{C}[\sigma^\vee] := \mathbb{C}[s_1^{i_1} \cdot \ldots \cdot s_N^{i_N} : (i_1, \ldots, i_N) \in \sigma^\vee \cap \mathbb{Z}^N]$$

will be finitely generated, hence its *spectrum* is an affine algebraic variety, called the *affine toric variety* associated to the cone σ.

Two smooth examples



$\mathbb{C}^2$



$\mathbb{C}^2$

Affine toric varieties of type $\mathbf{A_{n,q}}$

Given the pair (n, q), we call the affine toric variety associated to the cone $\sigma_{n,q}^{\vee} = \sigma_{n,q}^{c}$ the *toric variety of type* $A_{n,q}$ and denote it by the symbols $X_{n,q}$ or $X(a_1, \ldots, a_m)$.

Example

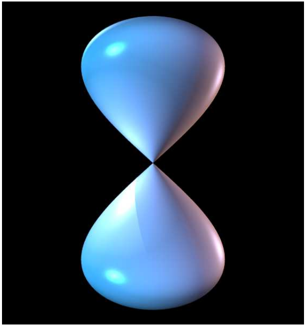
For $q := n - 1$, *one has* $n/(n-q) = n$. *Therefore, the ring of regular/holomorphic functions of the affine toric variety associated to the cone* $\sigma_{n,n-1}^{\vee}$ *will be generated by the monomials*

$$x_0 = s\,, \; x_1 = s\,t\,, \; x_2 = s^{n-1}\,t^n$$

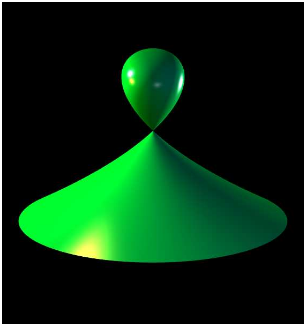
with the generating relation

$$x_0\,x_2 = x_1^n$$

such that singularities of type $A_{n,n-1}$ *are the* simple *singularities of type* A_{n-1}.

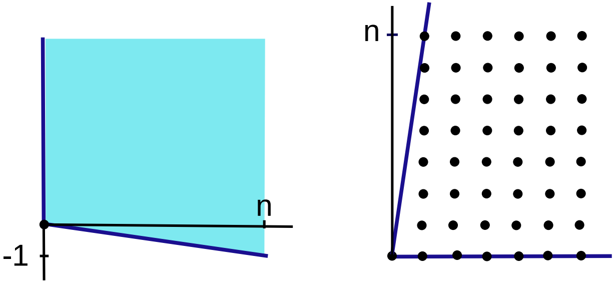


Octdong



Dingdong

The cone over the rational normal curve of degree n



Generating monomials: $x_0 := s,\ x_1 = st,\ x_2 := st^2, \ldots, x_n = st^n$.
Equations: Vanishing of all 2×2 - minors of the matrix

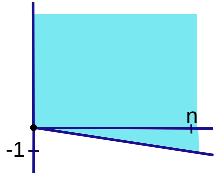
$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \end{pmatrix} .$$

The resolution of the cone singularities

The previous equations can be viewed as the projective equations of a *curve* in projective space $\mathbb{P}_n$; it is isomorphic to the projective line $\mathbb{P}_1$ under the Veronese - embedding with respect to the line bundle $\mathcal{O}_{\mathbb{P}_1}(n)$. The (minimal) resolution of the cone singularity is well known to being isomorphic to the total space of the dual bundle $\mathcal{O}_{\mathbb{P}_1}(-n)$.

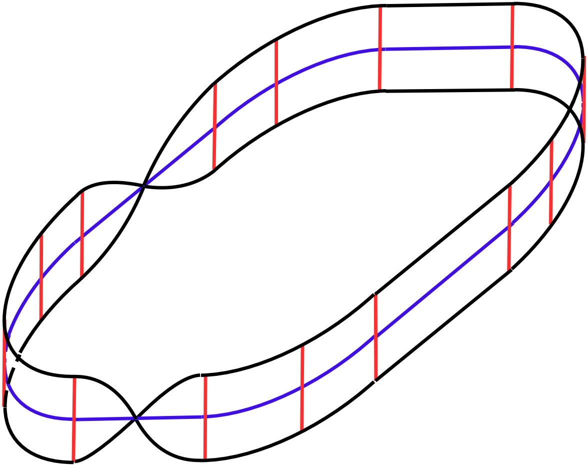
This can also be understood via toric geometry: One has to decompose the rational cone σ in such a manner into a *fan* such that the "subcones" are generated by $\mathbb{Z}$ - bases of $\mathbb{Z} \oplus \mathbb{Z}$ and the corresponding affine varieties are suitably patched together.

In our example this leads to the following picture and patching rule:

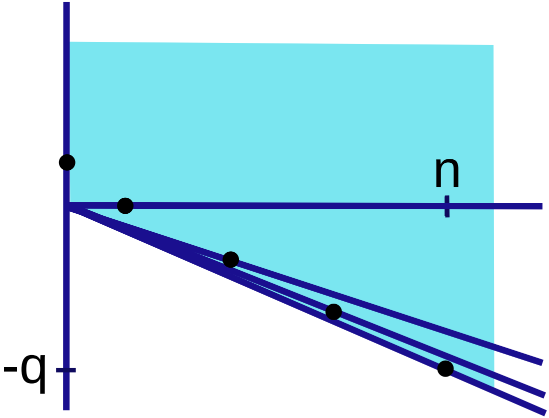


$$t = v^{-1}, \; s = w\, v^n$$

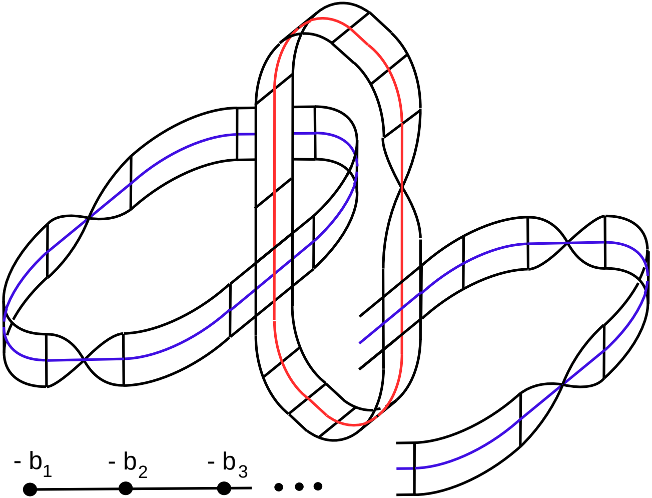
Schematic picture of the resolution



In the general case (n, q) one has to regard the following cone and its fan decomposition:



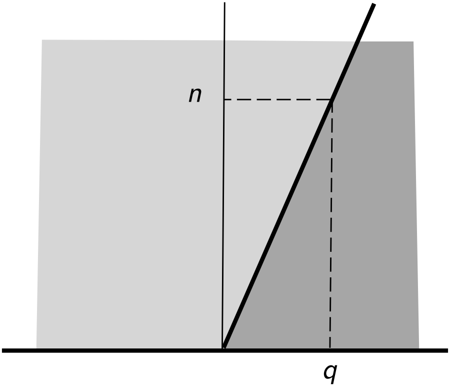
The resolution hence arises by gluing r holomorphic line bundles.


- b1
- b2
- b3

Toric Varieties

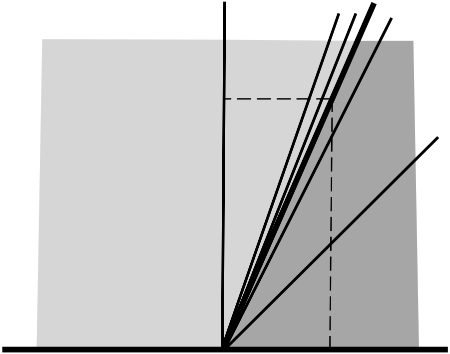
General *toric varieties* are constructed by gluing affine toric varieties; the underlying combinatorics is encoded again in fans.

Example: The fan for $\overline{X}_{n,q}$.

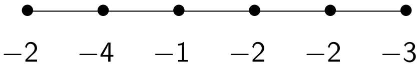


Resolution of $\overline{X}_{n,q}$

$$\frac{7}{3} = 3 - \frac{1}{2 - \frac{1}{2}}\,, \qquad \frac{7}{4} = 2 - \frac{1}{4}$$



This is necessarily a *modification* of the trivial line bundle $\mathbb{P}_1 \times \mathbb{C}$. More precisely, it is



In fact, one can blow this configuration down symbolically as follows:

$$
\begin{array}{ccccccccccc}
2 & & 4 & & 1 & & 2 & & 2 & & 3 \\
 & 2 & & 3 & & 1 & & 2 & & 3 & \\
 & & 2 & & 2 & & 1 & & 3 & & \\
 & & & 2 & & 1 & & 2 & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & & & 0 & & & & &
\end{array}
$$

4. Cyclic quotient surface singularities

Holomorphic operations of cyclic groups

Local quotients of $\mathbb{C}^N$ by a finite group Γ of local holomorphic automorphisms are - per definitionem - the spectrum of the invariant ring $\mathbb{C}\{u_1, \ldots, u_N\}^\Gamma$.

Local quotients of $\mathbb{C}^2$ by a finite *cyclic* group of holomorphic automorphisms are isomorphic to the quotients

$$\mathbb{C}^2 / \Gamma_{n,q} \ ,$$

where the group $\Gamma_{n,q}$ is generated by the diagonal matrix

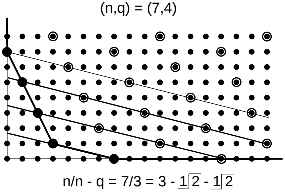
$$\mathrm{diag}\,(\zeta_n,\, \zeta_n^q)$$

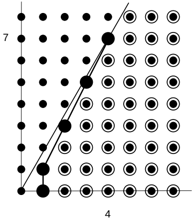
with an n-th primitive root of unity ζ_n and q with $1 \leq q < n$ relatively prime to n .

One has $\mathbb{C}^2 / \Gamma_{n,q} \cong \mathbb{C}^2 / \Gamma_{n',q'}$ if and only if $n = n'$ and $q = q'$ or $n = n'$ and $q\,q' \equiv 1 \bmod n$.

The invariant ring

The invariant ring $\mathbb{C}[u, v]^{\Gamma_{n,q}}$ consists obviously of all polynomials in the monomials $u^i v^j$ with $i + qj = kn$. The corresponding lattice can (after suitable lattice point transformation and normalization) be transferred to the lattice of the singularity $A_{n,q}$:





Corollary

$$\mathbb{C}^2 / \Gamma_{n,q} \cong X_{n,q}\,, \quad (\mathbb{P}_1 \times \mathbb{C}) / \Gamma_{n,q} \cong \overline{X}_{n,q}\,.$$

Putting with $n/(n-q) = a_1 - 1 \rfloor a_2 - \cdots - 1 \rfloor a_m$:

$$i_0 = n\,,\; i_1 = n - q\,,\; i_{\mu+1} = a_\mu\, i_\mu - i_{\mu-1}\,,$$
$$j_0 = 0\,,\; j_1 = 1 \qquad\quad ,\; j_{\mu+1} = a_\mu\, j_\mu - j_{\mu-1}\,,$$

this implies:

Theorem

The monomials

$$x_\mu = u^{i_\mu} v^{j_\mu}\,, \quad \mu = 0, \ldots, m+1$$

form a minimal generating system of the invariant ring referring to the group $\Gamma_{n,q}$.

In particular, the singularity $X_{n,q}$ *possesses the embedding - dimension*

$$e = m + 2\,.$$

Multiplicity and equations

As every *rational* singularity of embedding dimension e the singularity $X(a_1, \ldots, a_m)$ has the *multiplicity* $e - 1 = m + 1$ and a minimal generating system of equations of cardinality

$$\binom{e-1}{2} = \binom{m+1}{2} .$$

They can be written in *quasi - determinantal* form:

$$\begin{pmatrix} x_0 & & x_1 & & x_2 & \cdots & x_{m-1} & & x_m \\ & x_1^{a_1-2} & & x_2^{a_2-2} & & \cdots & & x_m^{a_m-2} & \\ x_1 & & x_2 & & x_3 & \cdots & x_m & & x_{m+1} \end{pmatrix} .$$

5. Deformations

Versal Deformations

Versal Deformations exist e. g. for

1. isolated singularities
2. resolutions of isolated singularities
3. compact complex spaces
4. holomorphic vector bundles on fixed base spaces

$$\begin{array}{c} \mathfrak{X}_{\text{vers}} \\ \Big\downarrow \\ S_{\text{vers}} \end{array}$$

$$\begin{array}{ccc} \mathfrak{X} & & \mathfrak{X}_{\mathrm{vers}} \\ \downarrow & & \downarrow \\ S & & S_{\mathrm{vers}} \end{array}$$

$$
\begin{tikzcd}
\mathfrak{X} \arrow[d] & \mathfrak{X}_{\mathrm{vers}} \arrow[d] \\
S \arrow[r, "\exists"'] & S_{\mathrm{vers}}
\end{tikzcd}
$$

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d] & \mathfrak{X}_{\mathrm{vers}} \arrow[d] \\ S \arrow[r] & S_{\mathrm{vers}} \end{tikzcd}$$

The Versal Deformation of the Resolution

The versal deformation $\widetilde{\mathfrak{X}} \to T_{\mathrm{vers}}$ of the resolution $\widetilde{X}_{n,q}$ can explicitly be written down where T_{vers} is a smooth base space of dimension

$$\dim\ T_{\mathrm{vers}} = \sum_{\rho=1}^{r} (b_\rho - 1)\,.$$

This deformation can fiberwise be *blown down* by a general theorem of WAHL and R. to a deformation of the given singularity (the "Artin deformation"), and due to a theorem of M. ARTIN and LIPMAN there exists a *smooth component* S_{art} of the versal base space S_{vers} (the "Artin component") such that the existing mapping

$$T_{\mathrm{vers}} \longrightarrow S_{\mathrm{vers}}$$

factorizes over S_{art}; more precisely $S_{\mathrm{art}} = T_{\mathrm{vers}}/W$.

The Artin component

In the case of a cyclic quotient singularity one can easily determine the total deformation over the Artin component as the "generic perturbations" of the quasi - determinantal formats of the given singularity $X\,(a_1, \ldots, a_m)$ over a smooth base space of dimension

$$\sum_{\mu=1}^{m} (a_\mu - 1) \ :$$

$$\begin{pmatrix} x_0 & & x_1 + t_1 & & x_2 + t_2 & \cdots & x_{m-1} + t_{m-1} & & x_m + t_m \\ & X_1^{(a_1-2)} & & X_2^{(a_2-2)} & & \cdots & & X_m^{(a_m-2)} & \\ x_1 & & x_2 & & x_3 & \cdots & x_m & & x_{m+1} \end{pmatrix}$$

with

$$X_\mu^{(a_\mu-2)} := x_\mu^{a_\mu-2} + s_\mu^{(1)} x_\mu^{a_\mu-3} + \cdots s_\mu^{(a_\mu-2)} \ .$$

The Artin deformation space over T_{vers} can be obtained by *linear factorization* of the polynomials $X_\mu^{(a_\mu - 2)}$. The Weyl - group W is hence nothing else but the product of symmetric groups

$$W = \prod_{\mu=1}^{m} \mathfrak{S}_{a_\mu - 2} \, .$$

Corollary

$$\sum_{\mu=1}^{m} (a_\mu - 1) = \sum_{\rho=1}^{r} (b_\rho - 1) \, .$$

Remark

The last identity can also be derived directly from the continued fraction expansions.

The dot - diagram

In case (47, 11) look at the following diagram:

$$
\begin{array}{cccccc}
5 & \bullet & \bullet & \bullet & \bullet & \\
2 & & & & \bullet & \\
2 & & & & \bullet & \\
3 & & & & \bullet & \bullet \\
2 & & & & & \bullet \\
 & 2 & 2 & 2 & 5 & 3
\end{array}
$$

From a similar diagram one can always read off the continued fraction expansion of $n/n-q$ from that of n/q without knowing n and q.

Further components

From general reasons one can conclude that the deformations over S_{art} are already versal for the small embedding dimensions 3 and 4. In case $e \geq 5$, i. e. $m \geq 3$, one knows, however, that

$$\begin{aligned} \dim\, T^1 = \sum_{\mu=1}^{m} a_\mu - 2 \quad &= \sum_{\mu=1}^{m} (a_\mu - 1) + (m - 2) \\ > \sum_{\mu=1}^{m} (a_\mu - 1)\,. & \end{aligned}$$

Consequently, there must more components of the base space of the versal deformation exist, *non - embedded* ones (in which case these components of the base space must be smooth) or even *embedded* ones.

Example

- *For* $e = 5$ *there exist always* 2 *"smooth" components and no embedded ones.*
- *For* $e = 6$ *there exist between one and five smooth components and embedded components, if the maximal number 5 is not attained.*

Questions:

- How many such components exist for a given singularity $X_{n,q} = X(a_1, \ldots, a_m)$?
- And how can one describe them?

The monodromy coverings

For the determination of the possible singularities in neighboring fibers in general deformations one needs only this information for the deformation over the reduction of $S_{\rm vers}$:

$$(\mathfrak{X}_{\rm vers})_{\rm red} \longrightarrow (S_{\rm vers})_{\rm red}$$

which, by abuse of notation, we abbreviate by

$$\mathfrak{X}_{\rm red} \longrightarrow S_{\rm red} \ .$$

It is well known (Christoffersen) that $S_{\rm red}$ has only *smooth* components S_j , and for each S_j there is a finite covering $T_j \to S_j$ such that the total space $\mathfrak{Y}_j$ of the induced deformation over T_j is equipped with a *toroidal* structure.

The monodromy coverings and the Artin component

The families $\mathfrak{Y}_j \to T_j$ have no *monodromy*, in particular very special *fiber singularities*.

In particular, there is exactly one component S_0 of S such that the family $\mathfrak{Y}_0 \to T_0$ can be *resolved simultaneously* (without base change): the Artin component.

Remark

One part of Martin Hamm*'s Hamburg dissertation (2008) is concerned with* toroidal *descriptions of the monodromy coverings* $\mathfrak{Y}_j \to T_j$ *and even of the* grand monodromy covering

$$\mathfrak{Y} \to T\ .$$

6. Duality for the Artin deformation

Duality for the Artin deformation

Theorem (M. Hamm)

Let X be the singularity $X_{n,q}$, and X' the singularity $X_{n,n-q}$. Then their total Artin deformation spaces are dual to each other as affine toric varieties.

In other words: If the Artin deformation space of X will be described by the pair $(\mathbb{L}, \Sigma)$, then the Artin deformation space of X' will be described by $(\mathbb{L}^*, \Sigma^\vee)$.

More precisely: There is a simultaneous way how to describe the toric generators for the algebra defining this total space in case $X_{n,q}$ with the help of the continued fraction expansion for $n/(n-q)$, i. e. the "a - series", and how to find the inequalities of the underlying rational convex cone (up to isomorphism) by the expansion for n/q, i. e. the "b - series".

The case A_1

The generators in case A_1 are given by the vectors

$$e_0,\ e_0 + e_1,\ e_0 + 2e_1 \in \mathbb{Z}^2 .$$

(In the following, we always denote by $e_0, \ldots, e_n$ the standard basis of $\mathbb{R}^{n+1}$). The four vectors

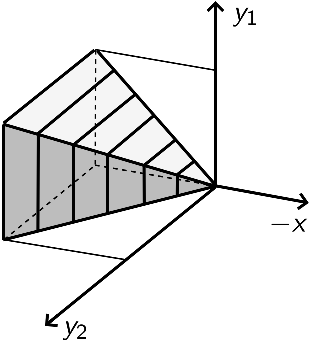
$$x_0 = e_0\,,\ x_1 = e_0 + e_1\,,\ x_1' = e_0 + e_2\,,\ x_2 = e_0 + e_1 + e_2 \in \mathbb{Z}^3$$

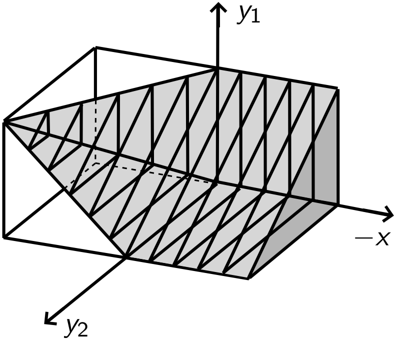
project down to these three vectors under the projection $e_2 \mapsto e_1$. They satisfy the unique generating relation

$$x_0 + x_2 = x_1 + x_1' .$$

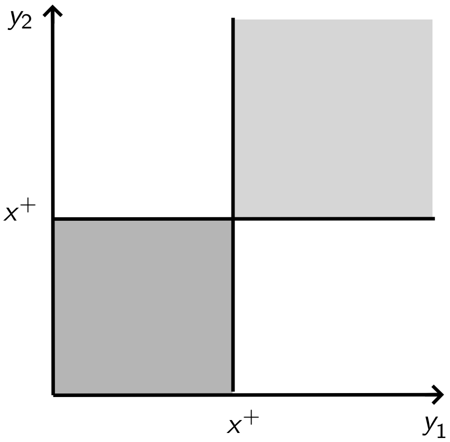
Therefore, this defines the toric generators for the Artin deformation in question, the projection corresponding to the embedding of the singularity as a fiber in this space.

It is easy to see that the generated cone will be described exactly by four inequalities leading to the dual cone on the right hand side. But this one is obviously the cone of the Artin deformation again if one starts a priori with σ^c instead of σ .



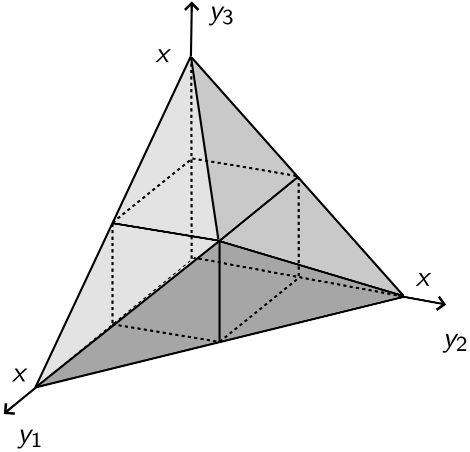


These two cones are also complements of each other in some sense as one can better imagine when taking slides only:



The case $\mathbf{A_2}$

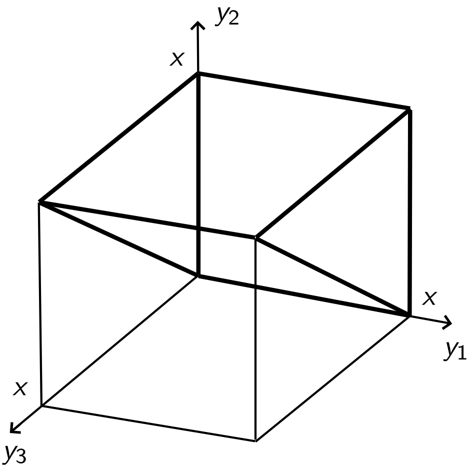
A manifestation of the Artin deformation for A_2 in $\mathbb{R}^4$ is a generalization of the above mentioned cone which, after isomorphism, can also be realized by the following picture.



For general n, the cone giving the Artin deformation of A_n should again be chosen with respect to the *complementary* cone in $\mathbb{R}^2$. Then, it is easy to determine the *dual* one:

$$0 \le y_1 \le x \,, \quad 0 \le y_n \le y_{n-1} \le \cdots \le y_2 \le x \,.$$

<u>Case</u> A_2:



Example (4,2,3)

After an obvious lattice transformation, the generators for the Artin deformation can be written in the following "format":

$$\begin{pmatrix} \mathbf{e}_1 & & & \mathbf{e}_4 & \mathbf{e}_5 & & \mathbf{e}_7 \\ & \mathbf{e}_2 & \mathbf{e}_3 & & & \mathbf{e}_6 & \\ \mathbf{e}_0 & & & \times & \times & & \times \end{pmatrix} .$$

The crosses have to be filled in in the "obvious correct" way. In this case, they have to be successively from left to right:

$$\begin{aligned} f_1 &= e_0 - e_1 + e_2 + e_3 + e_4 \,, \\ f_2 &= e_0 - e_1 + e_2 + e_3 + e_5 \,, \\ f_3 &= e_0 - e_1 + e_2 + e_3 + e_6 + e_7 \,. \end{aligned}$$

Example (2,2,4,2)

One can easily compute generators for the dual cone according to this format:

$$e_0\,,\ e_2\,,\ \ldots\,,\ e_7\,,\ e_0+e_1\,,\ e_1+e_2\,,\ e_1+e_3\,,$$

$$e_1+e_4+e_5+e_6\,,\ e_1+e_4+e_5+e_7\,.$$

After replacing e_1 by e_1-e_0 and then interchanging e_0 and e_1, the new generators fit exactly into the dual format:

$$\begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} & & & \mathbf{e_6} & \mathbf{e_7} \\ & & & \mathbf{e_4} & \mathbf{e_5} & & \\ \mathbf{e_0} & \times & \times & & & \times & \times \end{pmatrix}.$$

Remark

The action of the cyclic group $\Gamma_{n,q}$ *on* $\mathbb{C} \times \mathbb{C}$ *can uniquely be extended, as we have seen, to an action on the trivial bundle* $\mathbb{P}_1 \times \mathbb{C}$ *over* $\mathbb{P}_1$ *. The quotient* $\overline{X}_{n,q}$ *possesses a holomorphic mapping* $\psi : \overline{X}_{n,q} \to \mathbb{P}_1$ *with a singularity of type* $A_{n,q}$ *over the origin* 0 *resp. of type* $A_{n,n-q}$ *over* ∞ *.*

Conjecture

The Artin deformation of $\psi^{-1}(\mathbb{P}_1 \setminus \{\infty\})$ *can at* ∞ *be extended to the Artin deformation of* $\psi^{-1}(\mathbb{P}_1 \setminus \{0\})$ *.*

Dès que nous exprimons quelque chose, nous le diminuons étrangement. Nous croyons avoir plongé jusqu'au fond des abîmes et quand nous remontons à la surface, la goutte d'eau qui scintille au bout de nos doigts pâles ne ressemble plus à la mer d'où elle sort. Nous croyons avoir découvert une grotte aux trésors merveilleux; et quand nous revenons au jour, nous n'avons emporté que des pierreries fausses et des morceaux de verre; et cependent le trésor brille invariablement dans les ténèbres.

(Maurice Maeterlinck, *La Morale Mystique*, in: *Le trésor des humbles*)

How strangely do we diminish a thing as soon as we try to express it in words! We believe we have dived down to the most unfathomable depths, and when we reappear on the surface, the drop of water that glistens on our trembling finger-tips no longer resembles the sea from which it came. We believe we have discovered a grotto that is stored with bewildering treasure; we come back to the light of day, and the gems we have brought are false – mere pieces of glass – and yet does the treasure shine on, unceasingly, in the darkness!

(*Mystic Morality*, in: *The Treasure of the Humble*)
(Translation by Alfred Sutro)