A Study on Symmetric Quotients

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Bericht Nr. 1998-06

Dezember 1998

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Abstract

Symmetric quotients, introduced in the context of heterogeneous relation algebras, have proven useful for applications comprising for example program semantics and databases.

Recently, the increased interest in fuzzy relations has fostered a lot of work concerning relation-like structures with weaker axiomatisations.

In this paper, we study symmetric quotients in such settings and provide many new proofs for properties previously only shown in the strong theory of heterogeneous relation algebras. Thus we hope to make both the weaker axiomatisations and the many applications of symmetric quotients more accessible to people working on problems in some specific part of the wide spectrum of relation categories.

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1 Introduction

The symmetric quotient has been introduced in [BSZ86; BSZ89] as the intersection of two residuals in the context of heterogeneous relation algebras. For concrete relations R and S, the symmetric quotient relates elements r from the range of R with elements s from the range of S exactly if the inverse image of r under R is the same as the inverse image of s under S, or, in the language of predicate logic:

$$(r, s) \in \mathrm{syq}(R, S) \qquad \Longleftrightarrow \qquad \forall x : (x, r) \in R \leftrightarrow (x, s) \in S$$

(Riguet had introduced the unary operation of "noyeau", which can now be seen as defined by $\mathsf{noy}(R) = \mathrm{syq}(R, R)$, in [Rig48].) Heterogeneous relation algebras (see [SS93; BKS97]) somewhat abstract away from concrete binary relations between sets, so the formalisation of symmetric quotients of [BSZ86; BSZ89] makes that concept accessible also in *abstract* heterogeneous relation algebras.

In the literature, besides heterogeneous relation algebras also other, weaker formalisations of relation-like structures have been investigated. An especially fine-grained system of such formalisations are the various kinds of *allegories* of [FS90]. By using their tool box, one may aggregate monster names like "locally complete unitary pretabular allegory" to describe axiomatisations of carefully selected aspects of relation-like structures. Many of these are also studied independently in other sources and under different names.

For the concrete application to *fuzzy relations*, recently there has been increased interest in *Dedekind categories* (defined by [OS80] and equivalent to complete division allegories), witness [KFM96; Fur98].

Since the symmetric quotient is defined as the intersection of two residuals, obviously the setting of division allegories is already sufficient for being able to define symmetric quotients. In [BSZ86; BSZ89] however, since working in the context of heterogeneous relation algebras, the double-negation formulation of residuals is employed, and many properties of symmetric quotients have been shown using the Schröder equivalences and other properties of negation, and a few nice properties with respect to the interaction of symmetric quotients with negation could also be shown.

Of course, the presence of negation is not always a prerequisite.

In this paper, we therefore set out to make the many possible applications of symmetric quotients accessible to people working in other, weaker formalisations of relation-like structures. To this purpose we try to reformulate as much of the previous work on symmetric quotients as possible in the formalisms of division allegories and Dedekind categories, which lack negation, but still feature residuals.

This paper is organised as follows: In Sect. 2 we present as a starting point distributive allegories, which do not yet feature residuals.

The existence of residuals is then the common feature of all the different relation categories presented in Sect. 3, starting from division allegories and finishing with heterogeneous abstract relation algebras.

Section 4 lists useful properties of residuals, some perhaps new, but mostly taken from other sources, although some of those required more general proofs.

A novel feature presented in this paper is the consideration of symmetric quotients in the context of distributive allegories, i.e., without assuming the existence of residuals, in Sect. 5 — some of the useful properties of symmetric quotients already hold in this very weak setting.

In Sect. 6, symmetric quotients are considered in the presence of residuals, and many properties originally only shown in heterogeneous relation algebras and using negation and Schröder equivalences are provided with new proofs only using properties of division allegories or Dedekind categories as appropriate.

This task is continued in the remaining sections, where we reformulate the interplay of symmetric quotients with vectors and points (Sect. 7) and orderings (Sect. 8), and the use of symmetric quotient towards relational specification of sets in Sect. 9.

Besides the list of references, the appendix also includes an index and a few auxiliary properties of relations.

Throughout this paper, we adhere to the notation for abstract relation algebras fixed in [BKS97].

Definitions, lemmata, etc. include forward-references to the pages of their use in the shape "$[\leftarrow p_1, \dots, p_n]$".

2 Allegory Preliminaries

Although Freyd and Scedrov start their considerations of relation categories with *allegories* that extend categories only with conversion and intersection of relation-morphisms, we jump right to the next level that also includes empty relations and union of relations.

Definition 2.1 A **distributive allegory** is a category $\mathcal{D}$ consisting of a class $\mathsf{Obj}_{\mathcal{D}}$ of objects and a set[1] $\mathsf{Mor}_{\mathcal{D}}[A,B]$ of morphisms for all $A, B \in \mathsf{Obj}_{\mathcal{D}}$. The morphisms are usually called *relations*. For $S \in \mathsf{Mor}_{\mathcal{D}}[A,B]$ we use the notation $S : A \leftrightarrow B$. Composition is denoted by ";" and identities by $\mathbb{I}_A : A \leftrightarrow A$. In addition, there is the total unary operation $\breve{}$ of *conversion* of morphisms, where for $R : A \leftrightarrow B$ we have $R\breve{} : B \leftrightarrow A$. The operations satisfy the following rules:

i) Every set $\mathsf{Mor}_{\mathcal{D}}[A,B]$ carries the structure of a distributive lattice with operations $\sqcup_{A,B}$ for *join*, $\sqcap_{A,B}$ for *meet*, zero element $\mathbb{⊥}_{A,B}$, and inclusion ordering $\sqsubseteq_{A,B}$, all usually written without indices.
ii) The conversion is a monotone and involutive contravariant functor:
   (a) $(R\breve{})\breve{} = R$,
   (b) $(Q;R)\breve{} = R\breve{};Q\breve{}$,
   (c) $(Q \sqcap Q')\breve{} = Q\breve{} \sqcap Q'\breve{}$.
iii) For all $Q : A \leftrightarrow B$ and $R, R' : B \leftrightarrow C$, *meet-subdistributivity* and *join-distributivity* hold:

$$Q;(R \sqcap R') \sqsubseteq Q;R \sqcap Q;R' \qquad \text{and} \qquad Q;(R \sqcup R') = Q;R \sqcup Q;R' \ .$$

[1] This may be a class in [FS90], meaning that there, allegories are not restricted to be locally small. The price of this generality, however, is that join, meet, etc. need to be characterised at a more elementary level, while we can introduce these as lattice operators. We therefore sacrifice that generality for the sake of brevity and readability.

iv) For all $Q : A \leftrightarrow B$, the *zero law* holds:

$$Q ; \perp\!\!\!\perp_{B,C} = \perp\!\!\!\perp_{A,C} \ .$$

v) For all $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$, and $S : A \leftrightarrow C$, the *modal rule* holds:

$$Q ; R \sqcap S \sqsubseteq (Q \sqcap S ; R^{\smile}) ; R \ .$$

□

The following basic properties are easily deduced from the definition of distributive allegories:

Proposition 2.2 Let $Q, Q' : A \leftrightarrow B$ and $R, R' : B \leftrightarrow C$ be relations in $\mathcal{D}$. Then:

i) $\perp\!\!\!\perp^{\smile}_{A,B} = \perp\!\!\!\perp_{B,A}$ and $\mathbb{I}^{\smile}_{A} = \mathbb{I}_A$.

ii) If $Q \sqsubseteq Q'$ and $R \sqsubseteq R'$, then $Q ; R \sqsubseteq Q' ; R'$.

iii) If $Q \sqsubseteq Q'$, then $Q^{\smile} \sqsubseteq Q'^{\smile}$.

iv) $(Q \sqcup Q')^{\smile} = Q^{\smile} \sqcup Q'^{\smile}$.

v) $\perp\!\!\!\perp_{A,B} ; R = \perp\!\!\!\perp_{A,C}$. □

From the modal rule listed among the allegory axioms, we may — using properties of conversion — obtain the other modal rule

$$Q ; RS \sqsubseteq Q ;(R \sqcap Q^{\smile} ; S) \ ,$$

which is called Dedekind formula by Olivier and Serrato and used for their axiomatisation of Dedekind categories [OS80; OS95], see also the next section.

Proposition 2.3 Both *modal rules*

$$\begin{array}{rcll} Q ; R \sqcap S & \sqsubseteq & Q ;(R \sqcap Q^{\smile} ; S) & (m1) \\ Q ; R \sqcap S & \sqsubseteq & (Q \sqcap S ; R^{\smile}) ; R & (m2) \end{array}$$

together are equivalent to the *Dedekind rule*

$$Q ; R \sqcap S \sqsubseteq (Q \sqcap S ; R^{\smile}) ;(R \sqcap Q^{\smile} ; S) \ .$$

Proof: The modal rules follow immediately from the Dedekind rule:

$$Q ; R \sqcap S \sqsubseteq (Q \sqcap S ; R^{\smile}) ;(R \sqcap Q^{\smile} ; S) \sqsubseteq \left\{ \begin{array}{l} (Q \sqcap S ; R^{\smile}) ; R \\ Q ;(R \sqcap Q^{\smile} ; S) \end{array} \right.$$

Conversely, assume that the modal rules hold. Then we have

$$\begin{array}{rcll} Q ; R \sqcap S & \sqsubseteq & Q ;(R \sqcap Q^{\smile} ; S) \sqcap S & \text{(m1)} \\ & \sqsubseteq & (Q \sqcap S ;(R \sqcap Q^{\smile} ; S)^{\smile}) ;(R \sqcap Q^{\smile} ; S) & \text{(m2)} \\ & \sqsubseteq & (Q \sqcap S ; R^{\smile}) ;(R \sqcap Q^{\smile} ; S) \ . & \forall U, V : U \sqcap V \sqsubseteq V \end{array}$$

□

3 Dedekind Categories and Other Relation Categories

Building on the distributive allegories of the last section, we now present a spectrum of relation categories featuring *residuals*, also called *division* operators.

We start with the raw division allegories of Freyd and Scedrov, and move on via the Dedekind categories and Schröder categories of Olivier and Serrato to the original home of the symmetric quotients of [BSZ86; BSZ89], namely the heterogeneous relation algebras of Schmidt and Ströhlein.

Definition 3.1 [FS90] A *division allegory* is a distributive allegory $\mathcal{D}$ where for arbitrary relations $S : A \leftrightarrow C$ and $R : B \leftrightarrow C$, the *left residual* S/R defined by

$$Q ; R \sqsubseteq S \iff Q \sqsubseteq S/R \qquad \text{for all } Q : A \leftrightarrow B$$

exists. □

The conditions of meet-subdistributivity, join-distributivity and zero law listed for distributive allegories are not required in the axiomatisation of division allegories, since here they can be deduced using the residual.

Independent of Freyd and Scedrov, Olivier and Serrato defined a kind of relation categories in [OS80] which differs from division allegories precisely by being what is called "locally complete" in [FS90, 2.22]:

Definition 3.2 [OS80] A *Dedekind category* is a division allegory $\mathcal{D}$ where every homset $\mathsf{Mor}_{\mathcal{D}}[A, B]$ is a *complete* lattice with greatest element $\mathbb{T}_{A,B}$, called *universal relation.* □

In contrast to [FS90, 2.22], the infinite variants of meet-subdistributivity and join-distributivity, which form part of the definition of local completeness, need not be listed here, since they follow from the complete lattice structure via the presence of residuals — on the other hand, the full definition of local completeness implies the existence of residuals [FS90, 2.315], such that a Dedekind category is just a locally complete distributive allegory.

With respect to universal relations, we have the following simple properties:

Proposition 3.3 [←16] Let A and B be two objects of a Dedekind category, then:

i) $\mathbb{T}_{A,B}^{\smile} = \mathbb{T}_{B,A}$.

ii) $\mathbb{T}_{A,A} ; \mathbb{T}_{A,B} = \mathbb{T}_{A,B} ; \mathbb{T}_{B,B} = \mathbb{T}_{A,B}$. □

Since this is less than what one is used to expect from concrete relations (and also from heterogeneous relation algebras), we define:

Definition 3.4 A Dedekind category has the property of *uniformity* iff for all objects A, B, C, composition of universal relations again yields a universal relation:

$$\mathbb{T}_{A,B} ; \mathbb{T}_{B,C} = \mathbb{T}_{A,C}$$ □

If all morphisms of a Dedekind category have complements, the Dedekind category is equivalent to a Schröder category:

Definition 3.5 A **Schröder category** [OS80; Jón88] is a Dedekind category where every homset is a Boolean lattice. □

The complement of a relation R is written $\overline{R}$.

It is well-known that in an allegory with Boolean lattices as homsets, the Dedekind rule is equivalent to the *Schröder equivalences*:

$$Q;R \sqsubseteq S \qquad\iff\qquad Q\breve{}\,;\overline{S} \sqsubseteq \overline{R} \qquad\iff\qquad \overline{S};R\breve{} \sqsubseteq \overline{Q}$$

for all relations $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$ and $S : A \leftrightarrow C$. For the first direction, it is sufficient to show that with the Dedekind rule, $Q;R \sqsubseteq S$ implies $Q\breve{}\,;\overline{S} \sqsubseteq \overline{R}$: assume $Q;R \sqsubseteq S$, then that is equivalent to $Q;R \sqcap \overline{S} = \bot$, and we have

$$Q\breve{}\,;\overline{S} \sqcap R \sqsubseteq Q\breve{}\,;(\overline{S} \sqcap Q;R) = \bot \ .$$

Conversely, assume that the Schröder equivalences hold. Then [SS85] shows:

$$\begin{array}{lll}
 & Q;R & \\
= & ((Q \sqcap S;R\breve{}) \sqcup (Q \sqcap \overline{S;R\breve{}}));((R \sqcap Q\breve{}\,;S) \sqcup (R \sqcap \overline{Q\breve{}\,;S})) & \text{Boolean lattice} \\
= & (Q \sqcap S;R\breve{});(R \sqcap Q\breve{}\,;S) \sqcup (Q \sqcap S;R\breve{});(R \sqcap \overline{Q\breve{}\,;S}) & \\
 & \sqcup\ (Q \sqcap \overline{S;R\breve{}});(R \sqcap Q\breve{}\,;S) \sqcup (Q \sqcap \overline{S;R\breve{}});(R \sqcap \overline{Q\breve{}\,;S}) & \text{join-distributivity} \\
\sqsubseteq & (Q \sqcap S;R\breve{});(R \sqcap Q\breve{}\,;S) \sqcup Q;\overline{Q\breve{}\,;S} \sqcup \overline{S;R\breve{}};R & \forall U, V : U \sqcap V \sqsubseteq U \\
\sqsubseteq & (Q \sqcap S;R\breve{});(R \sqcap Q\breve{}\,;S) \sqcup \overline{S} & \text{Schröder,}
\end{array}$$

yielding the Dedekind rule $Q;R \sqcap S \sqsubseteq (Q \sqcap S;R\breve{});(R \sqcap Q\breve{}\,;S)$.

Furthermore, the Schröder equivalences allow us to calculate:

$$Q;R \sqsubseteq S \qquad\iff\qquad \overline{S};R\breve{} \sqsubseteq \overline{Q} \qquad\iff\qquad Q \sqsubseteq \overline{\overline{S};R\breve{}}$$

Therefore, we have $S/R = \overline{\overline{S};R\breve{}}$, so that in Schröder categories the residual is defined *a priori* and need not be listed in the axiomatisation.

Finally, the usual definition of relation algebras most notably contains atomicity of the lattice structure of the homsets:

Definition 3.6 A **heterogeneous relation algebra** [SS93] is a Schröder category where every homset is an *atomic* and complete Boolean lattice with $\top \neq \bot$ and the Tarski rule

$$R \neq \bot_{A,B} \iff \top_{C,A};R;\top_{B,D} = \top_{C,D}$$

holds for all $R \in \mathsf{Mor}_{\mathcal{R}}[A,B]$ and $C, D \in \mathsf{Obj}_{\mathcal{R}}$. □

The Tarski rule, however, sometimes is dropped; if it is present, it ensures uniformity.

4 Properties of Residuals

The left residual S/R of two relations $S : A \leftrightarrow C$ and $R : B \leftrightarrow C$, defined (as above) by

$$Q ; R \sqsubseteq S \iff Q \sqsubseteq S/R \qquad \text{for all } Q : A \leftrightarrow B,$$

is called *right division* in [FS90, 2.312].

Dually to the left residual, obviously there also is a notion of the *right residual* $Q\backslash S : B \leftrightarrow C$ of given two relations $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ defined by

$$Q ; R \sqsubseteq S \iff R \sqsubseteq Q\backslash S \qquad \text{for all } R : B \leftrightarrow C.$$

This right residual is called *left division* in [FS90, 2.312] (but the symbols coincide with ours for both residuals).

For relations on sets, we have the following equivalences:

$$\begin{array}{rcl} x(Q\backslash S)y & \iff & \forall z : (zQx \Rightarrow zSy) \\ x(S/R)y & \iff & \forall z : (yRz \Rightarrow xSz) \end{array}$$

Proposition 4.1 Dedekind categories have right residuals.

Proof: Assume $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$ and $S : A \leftrightarrow C$. Then $Q\backslash S = (S^\smile/Q^\smile)^\smile$ follows via:

$$\begin{array}{rlclcl} R \sqsubseteq Q\backslash S & \iff & Q ; R \sqsubseteq S & \iff & (Q ; R)^\smile \sqsubseteq S^\smile & \\ & \iff & R^\smile ; Q^\smile \sqsubseteq S^\smile & \iff & R^\smile \sqsubseteq S^\smile/Q^\smile & \iff \quad R \sqsubseteq (S^\smile/Q^\smile)^\smile . \end{array} \qquad \square$$

Corollary 4.2 [←14] For relations $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$, and $S : A \leftrightarrow C$ we have:

$$(S/R)^\smile = R^\smile\backslash S^\smile \qquad \text{and} \qquad (Q\backslash S)^\smile = S^\smile/Q^\smile . \qquad \square$$

Freyd and Scedrov provide an alternative characterisation of the left residual via three inclusions [FS90, p. 225], and we also list the dual inclusions for the right residual:

Proposition 4.3 The residuals can equivalently be characterised by the following inclusion axioms:

$$\begin{array}{rclcrcl} (R_1 \sqcap R_2)/S & \sqsubseteq & R_1/S \sqcap R_2/S & \qquad & S\backslash(R_1 \sqcap R_2) & \sqsubseteq & S\backslash R_1 \sqcap S\backslash R_2 \\ T & \sqsubseteq & (T ; S)/S & & T & \sqsubseteq & S\backslash(S ; T) \\ (R/S) ; S & \sqsubseteq & R & & S ;(S\backslash R) & \sqsubseteq & R . \end{array} \qquad \square$$

Next we recall a few properties of the residuals. Many of these can be found in [FS90]. Another source is [Ohk98] where useful results of Hoare and He [HH86] concerning left residuals are transferred into Dedekind categories. Although we always state both dual versions of the properties, we only provide proofs for the left-residual variant.

Proposition 4.4 Let $Q, Q' : A \leftrightarrow B$, $R, R' : B \leftrightarrow C$ and $S, S' : A \leftrightarrow C$ be relations in a division allegory. Then the following holds:

i) [←12, 20] $(S/R);(R/T) \sqsubseteq S/T$ and $(Q\backslash S);(S\backslash U) \sqsubseteq Q\backslash U$ for arbitrary relations $T : D \leftrightarrow C$ and $U : A \leftrightarrow D$ [FS90, 2.314].

ii) [←18] $S/R \sqsubseteq (S;T)/(R;T)$ and $Q\backslash S \sqsubseteq (U;Q)\backslash(U;S)$ for arbitrary relations $T : C \leftrightarrow D$ and $U : D \leftrightarrow A$.

iii) [←9, 18, 23] If $S \sqsubseteq S'$, $R' \sqsubseteq R$ and $Q' \sqsubseteq Q$, then $S/R \sqsubseteq S'/R'$ and $Q\backslash S \sqsubseteq Q'\backslash S'$ [Ohk98].

Proof:

i) From $(S/R);(R/T);T \sqsubseteq (S/R);R \sqsubseteq S$ we obtain $(S/R);(R/T) \sqsubseteq S/T$.

ii) $Q \sqsubseteq S/R \iff Q;R \sqsubseteq S \Longrightarrow Q;R;T \sqsubseteq S;T \iff Q \sqsubseteq (S;T)/(R;T)$.

iii) For each $X : A \leftrightarrow B$ it holds that

$$X \sqsubseteq S/R \iff X;R \sqsubseteq S \Longrightarrow X;R' \sqsubseteq S' \iff X \sqsubseteq S'/R'$$

by the monotonicity of composition. □

From these, we may derive other useful properties:

Lemma 4.5 [←25] In division allegories, for $F : A \leftrightarrow B$, $R : B \leftrightarrow C$, $S : D \leftrightarrow C$, $U : A \leftrightarrow B$, $Q : A \leftrightarrow C$, $T : C \leftrightarrow D$ we have:

i) [←16] $F;(R/S) \sqsubseteq (F;R)/S$ and $(U\backslash Q);T \sqsubseteq U\backslash(Q;T)$.

ii) [←25] if F resp. $T^{\smile}$ are mappings, then the inclusions in i) are equalities.

Proof:

i) $F;(R/S) \sqsubseteq ((F;R)/R);(R/S) \sqsubseteq (F;R)/S$

ii) $(F;R)/S \sqsubseteq F;F^{\smile};((F;R)/S) \sqsubseteq F;((F^{\smile};F;R)/S) \sqsubseteq F;(R/S)$ □

With respect to identities, the following simple laws hold:

Proposition 4.6 Let $S, S' : A \leftrightarrow C$ be relations in a division allegory, then:

i) $\mathbb{I}_A \sqsubseteq S/S' \iff S' \sqsubseteq S \iff \mathbb{I}_C \sqsubseteq S'\backslash S$ [Ohk98].

ii) [←9, 20] $\mathbb{I}_A \sqsubseteq S/S$ and $\mathbb{I}_C \sqsubseteq S\backslash S$ [FS90, 2.314].

iii) $S = S/\mathbb{I}_C$ and $S = \mathbb{I}_A\backslash S$ [FS90, 2.314].

Proof: i) and ii) follow from the unit laws of identity.

iii) With Prop. 4.3 we obtain $S \sqsubseteq (S;\mathbb{I}_C)/\mathbb{I}_C = S/\mathbb{I}_C = (S/\mathbb{I}_C);\mathbb{I}_C \sqsubseteq S$. □

Note that for each relation $S : A \leftrightarrow C$, $(S/S);S = S = S;(S\backslash S)$ holds by (ii) and Prop. 4.3.

Proposition 4.7 For every relation $S : A \leftrightarrow C$ in a Dedekind category, the following hold:

i) $S/\mathbb{T}_{C,C} \sqsubseteq S$ and $\mathbb{T}_{A,A}\backslash S \sqsubseteq S$ [Ohk98].

ii) [←17, 22] $S/\perp\!\!\!\perp_{B,C} = \mathbb{T}_{A,B}$ and $\perp\!\!\!\perp_{A,B}\backslash S = \mathbb{T}_{B,C}$.

iii) [←16] $S\backslash\mathbb{T}_{A,B} = \mathbb{T}_{C,B}$ and $\mathbb{T}_{B,C}/S = \mathbb{T}_{B,A}$.

Proof:

i) follows from $S \sqsubseteq S;\mathbb{T}_{C,C}$ since composition is monotonic.

ii) $\mathbb{T}_{A,B} \sqsubseteq S/\perp\!\!\!\perp_{B,C} \iff \mathbb{T}_{A,B};\perp\!\!\!\perp_{B,C} \sqsubseteq S \iff \perp\!\!\!\perp_{A,C} \sqsubseteq S$

iii) $X \sqsubseteq S\backslash\mathbb{T}_{A,B} \iff S;X \sqsubseteq \mathbb{T}_{A,B}$ □

Lemma 4.8 [←14, 23] Let $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$ and $S : A \leftrightarrow C$ be relations in a division allegory.

i) If $G : D \leftrightarrow B$ is univalent, then $G;(Q\backslash S) \sqsubseteq (Q;G^{\smile})\backslash S$ and $(S/R);G^{\smile} \sqsubseteq S/(G;R)$. If G is a mapping, then equality holds.

ii) If Q is injective and R is univalent, then $S;R^{\smile} \sqsubseteq S/R$ and $Q^{\smile};S \sqsubseteq Q\backslash S$.

Proof:

i) For univalent G we have: $Q;G^{\smile};G;(Q\backslash S) \sqsubseteq Q;(Q\backslash S) \sqsubseteq S$. If G is a mapping, then we have the following for arbitrary $X : D \leftrightarrow C$:

$$X \sqsubseteq (Q;G^{\smile})\backslash S \iff Q;G^{\smile};X \sqsubseteq S \iff G^{\smile};X \sqsubseteq Q\backslash S \iff X \sqsubseteq G;(Q\backslash S) \ .$$

ii) If R is univalent, $S;R^{\smile};R \sqsubseteq S \iff S;R^{\smile} \sqsubseteq S/R$. If Q is injective, $Q;Q^{\smile};S \sqsubseteq \mathbb{I}_A;S = S \iff Q^{\smile};S \sqsubseteq Q\backslash S$. □

Later we shall need the following simple interactions between different residuals:

Lemma 4.9 [←15] Let $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$ and $S : A \leftrightarrow C$ be given relations. Then the following holds:

i) $S/R = (S/S)\backslash(S/R)$ and $Q\backslash S = (Q\backslash S)/(S\backslash S)$.

ii) $S/R \sqsubseteq (R/S)\backslash(R/R)$ and $Q\backslash S \sqsubseteq (Q\backslash Q)/(S\backslash Q)$.

Proof:

i) $(S/S)\backslash(S/R) \sqsubseteq S/R$ is trivial by Prop. 4.6.ii) and Prop. 4.4.iii). Conversely, for arbitrary $X : A \leftrightarrow B$ we have:

$$X \sqsubseteq S/R \iff X;R \sqsubseteq S \Longrightarrow (S/S);X;R \sqsubseteq (S/S);S \ .$$

Now $(S/S);S \sqsubseteq S$ holds by Prop. 4.3. Then

$$(S/S);X;R \sqsubseteq S \iff (S/S);X; \sqsubseteq S/R \iff X \sqsubseteq (S/S)\backslash(S/R) \ .$$

ii) For arbitrary $X : A \leftrightarrow B$ we have:

$$X \sqsubseteq S/R \iff X;R \sqsubseteq S \Longrightarrow (R/S);X;R \sqsubseteq (R/S);S \ .$$

Since $(R/S);S \sqsubseteq R$ holds by Prop. 4.3, we get:

$$(R/S);X;R \sqsubseteq R \iff (R/S);X \sqsubseteq R/R \iff X \sqsubseteq (R/S)\backslash(R/R) \ .$$ □

5 Symmetric Quotients in Distributive Allegories

In heterogeneous relation algebras, the notion of symmetric quotient has been defined [BSZ86; BSZ89; SS93] by

$$\mathrm{syq}(Q,S) = \overline{Q^{\smile};\overline{S}} \sqcap \overline{\overline{Q^{\smile}};S}$$

for arbitrary relations $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$. In heterogeneous relation algebras, of course the following equations hold:

$$\begin{array}{rcl} \mathrm{syq}(Q,S) & = & \overline{Q^{\smile};\overline{S}} \sqcap \overline{\overline{Q^{\smile}};S} \;=\; Q\backslash S \sqcap Q^{\smile}/S^{\smile} \;=\; Q\backslash S \sqcap (S\backslash Q)^{\smile} \\ & = & \sup\{X \mid Q;X \sqsubseteq S \text{ and } X;S^{\smile} \sqsubseteq Q^{\smile}\} \ . \end{array}$$

Therefore the symmetric quotient as introduced above is — modulo conversion of the arguments — exactly the *symmetric division* as introduced by Freyd and Scedrov for division allegories [FS90, 2.35].

For concrete binary relations R and S between sets, we recall from the introduction that the $\mathrm{syq}(R,S)$ relates elements from the range of R with elements from the range of S exactly iff the inverse images are equal, or:

$$(r,s) \in \mathrm{syq}(R,S) \qquad \Longleftrightarrow \qquad \forall x : (x,r) \in R \leftrightarrow (x,s) \in S$$

Following a suggestion of Yasuo Kawahara, we first investigate symmetric quotients without assuming residuals — in this setting, $\mathrm{syq}(\_,\_)$ may be a partial operation:

Definition 5.1 [←11] In a distributive allegory, the **symmetric quotient** $\mathrm{syq}(Q,S) : B \leftrightarrow C$ of two relations $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ is defined by

$$X \sqsubseteq \mathrm{syq}(Q,S) \iff Q;X \sqsubseteq S \text{ and } X;S^{\smile} \sqsubseteq Q^{\smile} \qquad \text{for all } X : B \leftrightarrow C \ . \qquad \square$$

Lemma 5.2 The following properties hold for relations $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ if the symmetric quotients exist:

i) [←13, 14, 18, 21] $\mathrm{syq}(Q,S) = \mathrm{syq}(S,Q)^{\smile}$

ii) [←12, 13] $Q;\mathrm{syq}(Q,S) \sqsubseteq S$ and $\mathrm{syq}(Q,S);S^{\smile} \sqsubseteq Q^{\smile}$

iii) [←13] $\mathbb{I}_B \sqsubseteq \mathrm{syq}(Q,Q)$

iv) [←13] $Q;\mathrm{syq}(Q,Q) = Q$ and $Q^{\smile} = \mathrm{syq}(Q,Q);Q^{\smile}$.

v) If Q is univalent and surjective, then $\mathbb{I}_B = \mathrm{syq}(Q,Q)$.

Proof:

$$\begin{array}{rll} \text{i) } X \sqsubseteq \mathrm{syq}(Q,S) & \iff & Q;X \sqsubseteq S \text{ and } X;S^{\smile} \sqsubseteq Q^{\smile} \\ & \iff & X^{\smile};Q^{\smile} \sqsubseteq S^{\smile} \text{ and } S;X^{\smile} \sqsubseteq Q \\ & \iff & S;X^{\smile} \sqsubseteq Q \text{ and } X^{\smile};Q^{\smile} \sqsubseteq S^{\smile} \\ & \iff & X^{\smile} \sqsubseteq \mathrm{syq}(S,Q) \end{array}$$

ii) From Def. 5.1 via substituting $\mathrm{syq}(Q,S)$ for X.

iii) From Def. 5.1 via substituting $\mathbb{I}$ for X and Q for S.

iv) $Q;\mathrm{syq}(Q,Q) \sqsubseteq Q$ follows from ii). Conversely $Q = Q;\mathbb{I}_B \sqsubseteq Q;\mathrm{syq}(Q,Q)$ by iii).

v) Since Q is univalent and surjective, with iv) we obtain

$$\mathrm{syq}(Q,Q) = \mathbb{I}_B;\mathrm{syq}(Q,Q) = Q^{\smile};Q;\mathrm{syq}(Q,Q) = \mathbb{I}_B \ . \qquad \square$$

Proposition 5.3 [←21] In distributive allegories, the following properties hold for relations $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$:

i) $\mathrm{syq}(Q,S) \sqsubseteq \mathrm{syq}(P;Q,P;S)$ for each relation $P : D \leftrightarrow A$.

ii) $F;\mathrm{syq}(Q,S) \sqsubseteq \mathrm{syq}(Q;F^{\smile},S)$ if $F : D \leftrightarrow B$ is univalent; if F is a mapping, then equality holds.

iii) $\mathrm{syq}(Q,S);R = \mathrm{syq}(Q,S;R)$ for each injective and surjective $R : C \leftrightarrow D$.

Proof:

i)
$$\begin{array}{ll}
 & X \sqsubseteq \mathrm{syq}(Q,S) \\
\Longleftrightarrow & Q;X \sqsubseteq S \ \text{ and } \ X;S^{\smile} \sqsubseteq Q^{\smile} \\
\Longrightarrow & P;Q;X \sqsubseteq P;S \ \text{ and } \ X;S^{\smile};P^{\smile} \sqsubseteq Q^{\smile};P^{\smile} \\
\Longleftrightarrow & X \sqsubseteq \mathrm{syq}(P;Q,P;S)
\end{array}$$

ii)
$$\begin{array}{lll}
 & X \sqsubseteq F;\mathrm{syq}(Q,S) & \\
\Longrightarrow & Q;F^{\smile};X \sqsubseteq Q;F^{\smile};F;\mathrm{syq}(Q,S) \ \text{ and } \ X;S^{\smile} \sqsubseteq F;\mathrm{syq}(Q,S);S^{\smile} & \\
\Longrightarrow & Q;F^{\smile};X \sqsubseteq Q;\mathrm{syq}(Q,S) \ \text{ and } \ X;S^{\smile} \sqsubseteq F;Q^{\smile} & F \text{ univalent} \\
\Longrightarrow & Q;F^{\smile};X \sqsubseteq S \ \text{ and } \ X;S^{\smile} \sqsubseteq F;Q^{\smile} & \\
\Longleftrightarrow & X \sqsubseteq \mathrm{syq}(Q;F^{\smile},S) & \\
\Longleftrightarrow & Q;F^{\smile};X \sqsubseteq S \ \text{ and } \ X;S^{\smile} \sqsubseteq F;Q^{\smile} & \\
\Longrightarrow & Q;F^{\smile};X \sqsubseteq S \ \text{ and } \ F^{\smile};X;S^{\smile} \sqsubseteq F^{\smile};F;Q^{\smile} & \\
\Longrightarrow & Q;F^{\smile};X \sqsubseteq S \ \text{ and } \ F^{\smile};X;S^{\smile} \sqsubseteq Q^{\smile} & F \text{ univalent} \\
\Longleftrightarrow & F^{\smile};X \sqsubseteq \mathrm{syq}(Q,S) & \\
\Longrightarrow & F;F^{\smile};X \sqsubseteq F;\mathrm{syq}(Q,S) & \\
\Longrightarrow & X \sqsubseteq F;\mathrm{syq}(Q,S) & F \text{ total}
\end{array}$$

iii) is dual to ii). $\square$

Proposition 5.4 In a distributive allegory, let relations $Q : A \leftrightarrow B$ and $S : A' \leftrightarrow C$ be given. For each injective and surjective mapping $T : A \leftrightarrow A'$ we then have:

$$\mathrm{syq}(Q,T;S) = \mathrm{syq}(T^{\smile};Q,S) \qquad \text{and} \qquad \mathrm{syq}(T;Q,S) = \mathrm{syq}(Q,T^{\smile};S) \ .$$

Proof: With $T^{\smile};T = \mathbb{I}_A$ and $T;T^{\smile} = \mathbb{I}_{A'}$ and i) we get:

$$\mathrm{syq}(Q,T;S) \sqsubseteq \mathrm{syq}(T^{\smile};Q,T^{\smile};T;S) = \mathrm{syq}(T^{\smile};Q,S)$$

and

$$\mathrm{syq}(T^{\smile};Q,S) \sqsubseteq \mathrm{syq}(T;T^{\smile};Q,T;S) = \mathrm{syq}(Q,T;S) \ . \qquad \square$$

6 Symmetric Quotients in Dedekind Categories

We now harness the additional power of Dedekind categories to obtain more useful results about symmetric quotients, many of which had before only been shown with the help of negation and the Schröder equivalences.

Theorem 6.1 [←14, 17] In division allegories the symmetric quotient always exists and for two relations $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ we have

$$\mathrm{syq}(Q, S) = Q\backslash S \sqcap Q^{\smile}/S^{\smile} \ . \qquad \square$$

Proposition 6.2 [←21, 23, 24] Let $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ be are given relations. Then the following holds:

i) $Q ; \mathrm{syq}(Q, S) = S \sqcap \mathbb{T}_{A,B} ; \mathrm{syq}(Q, S)$.

ii) If $\mathrm{syq}(Q, S)$ is surjective, then $Q ; \mathrm{syq}(Q, S) = S$.

Proof:

i) $Q ; \mathrm{syq}(Q, S) \sqsubseteq S$ holds by Lemma 5.2.ii), and $Q ; \mathrm{syq}(Q, S) \sqsubseteq \mathbb{T}_{A,B} ; \mathrm{syq}(Q, S)$ by the monotonicity of the composition. Conversely it follows from

$$\begin{array}{rcl} S \sqcap \mathbb{T}_{A,B} ; \mathrm{syq}(Q, S) & \sqsubseteq & (S ; \mathrm{syq}(Q, S)^{\smile} \sqcap \mathbb{T}_{A,B}) ; \mathrm{syq}(Q, S) \\ & = & S ; \mathrm{syq}(S, Q) ; \mathrm{syq}(Q, S) \\ & \sqsubseteq & Q ; \mathrm{syq}(Q, S) \end{array}$$

by Dedekind formula and Lemma 5.2.ii).

ii) Assume that $\mathrm{syq}(Q, S)$ is surjective. Then $\mathbb{T}_{A,B} ; \mathrm{syq}(Q, S) = \mathbb{T}_{A,C}$ by Prop. A.2 iv). Thus ii) holds by i). $\square$

Proposition 6.3 [←13, 16, 21] Let $Q : A \leftrightarrow B$, $S : A \leftrightarrow C$ and $U : A \leftrightarrow D$ be are given relations. Then the following holds:

i) $\mathrm{syq}(Q, S) ; \mathrm{syq}(S, U) = \mathrm{syq}(Q, U) \sqcap \mathrm{syq}(Q, S) ; \mathbb{T}_{C,D} = \mathrm{syq}(Q, U) \sqcap \mathbb{T}_{B,C} ; \mathrm{syq}(S, U)$.

ii) If $\mathrm{syq}(Q, S)$ is total or $\mathrm{syq}(S, U)$ surjective, then $\mathrm{syq}(Q, S) ; \mathrm{syq}(S, U) = \mathrm{syq}(Q, U)$.

Proof:

i) Monotonicity of composition yields $\mathrm{syq}(Q, S) ; \mathrm{syq}(S, U) \sqsubseteq \mathrm{syq}(Q, S) ; \mathbb{T}_{C,D}$, and Prop. 4.4.i) gives us

$$\begin{array}{rcl} \mathrm{syq}(Q, S) ; \mathrm{syq}(S, U) & = & (Q\backslash S \sqcap Q^{\smile}/S^{\smile}) ; (S\backslash U \sqcap S^{\smile}/U^{\smile}) \\ & \sqsubseteq & (Q\backslash S) ; (S\backslash U) \sqcap (Q^{\smile}/S^{\smile}) ; (S^{\smile}/U^{\smile}) \\ & \sqsubseteq & Q\backslash U \sqcap Q^{\smile}/U^{\smile} \\ & = & \mathrm{syq}(Q, U) \ . \end{array}$$

Thus we have $\mathrm{syq}(Q,S);\mathrm{syq}(S,U) \sqsubseteq \mathrm{syq}(Q,U) \sqcap \mathrm{syq}(Q,S);\mathbb{T}_{C,D}$. Conversely it holds that

$$\begin{array}{rcl} \mathrm{syq}(Q,U) \sqcap \mathrm{syq}(Q,S);\mathbb{T}_{C,D} & \sqsubseteq & \mathrm{syq}(Q,S);(\mathrm{syq}(Q,S)^{\smile};\mathrm{syq}(Q,U) \sqcap \mathbb{T}_{C,D}) \\ & = & \mathrm{syq}(Q,S);\mathrm{syq}(S,Q);\mathrm{syq}(Q,U) \\ & \sqsubseteq & \mathrm{syq}(Q,S);\mathrm{syq}(S,U) \ . \end{array}$$

ii) is shown by i) and Prop. A.2 iii) or iv). □

Corollary 6.4 [←13] Let $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ be given relations. Then the following holds:

i) [←18, 22, 23] $\mathrm{syq}(Q,S);\mathrm{syq}(Q,S)^{\smile} \sqsubseteq \mathrm{syq}(Q,Q)$.

ii) $\mathrm{syq}(Q,Q);\mathrm{syq}(Q,S) = \mathrm{syq}(Q,S)$.

iii) [←14] $\mathrm{syq}(Q,Q)$ is a equivalence relation.

Proof:

i) follows from replacing U with Q in Prop. 6.3 i).

ii) follows from replacing S with Q and U with S in Prop. 6.3 ii), since $\mathrm{syq}(Q,Q)$ is total (with Lemma 5.2.iii)).

iii) $\mathrm{syq}(Q,Q)$ is reflexive by Lemma 5.2.iii), symmetric by Lemma 5.2.i) and transitive by Lemma 5.2.ii). □

Generally, a relation $Q : A \leftrightarrow B$ satisfies $Q \sqsubseteq Q;Q^{\smile};Q$ by $Q = Q \sqcap \mathbb{T}_{A,B} \sqsubseteq Q;(\mathbb{I}_B \sqcap Q^{\smile};\mathbb{T}_{A,B}) \sqsubseteq Q;Q^{\smile};(Q \sqcap \mathbb{T}_{A,B}) = Q;Q^{\smile};Q$. A relation $Q : A \leftrightarrow B$ is called difunctional if $Q;Q^{\smile};Q \sqsubseteq Q$ [SS93]. Hence Q is difunctional if and only if $Q;Q^{\smile};Q = Q$.

Proposition 6.5 Let $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ be given relations. Then $\mathrm{syq}(Q,S)$ is difunctional.

Proof: Implied from Corr. 6.4 i) and ii). □

Proposition 6.6 Let $P : A \leftrightarrow A$ be a given relation. Then the following holds:

i) P is symmetric if and only if $\mathbb{I}_A \sqsubseteq \mathrm{syq}(P^{\smile},P)$.

ii) P is symmetric and transitive if and only if $P \sqsubseteq \mathrm{syq}(P^{\smile},P)$.

iii) P is an equivalence relation if and only if $P = \mathrm{syq}(P,P)$.

Proof:

i) Assume that P is symmetric. Then $P^{\smile} = P$ holds. Thus we have $\mathbb{I}_A \sqsubseteq \mathrm{syq}(P^{\smile},P)$ by Lemma 5.2.iv). Next assume that $\mathbb{I}_A \sqsubseteq \mathrm{syq}(P^{\smile},P)$. Then $\mathbb{I}_A \sqsubseteq P^{\smile}\backslash P$ holds. By the definition of right residual $P^{\smile} \sqsubseteq P$.

ii) Assume that P is symmetric and transitive. Then $\mathrm{syq}(P^{\smile}, P) = P\backslash P \sqcap P/P$ holds since P is symmetric. Also it holds that

$$P \sqsubseteq P/P \iff P;P \sqsubseteq P \iff P \sqsubseteq P\backslash P$$

since P is transitive. Therefore it holds that $P \sqsubseteq \mathrm{syq}(P^{\smile}, P)$. Next assume that $P \sqsubseteq \mathrm{syq}(P^{\smile}, P)$. Then $P^{\smile};P \sqsubseteq P$ and $P;P \sqsubseteq P^{\smile}$ hold. So we have

$$P \sqsubseteq P;P^{\smile};P \sqsubseteq P;P \sqsubseteq P^{\smile} \ .$$

Therefore it holds that $P^{\smile} \sqsubseteq P$ and $P;P \sqsubseteq P$.

iii) If $P = \mathrm{syq}(P, P)$, P is an equivalence relation by Corr. 6.4.iii). Next assume that P is an equivalence relation. Then, by ii), $P \sqsubseteq \mathrm{syq}(P, P)$ holds. Also by the reflexivity of P, it holds that $\mathrm{syq}(P, P) \sqsubseteq P;\mathrm{syq}(P, P) = P$. □

Proposition 6.7 Let both $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ be given injective relations. Then the following holds:

i) $Q^{\smile};S \sqsubseteq \mathrm{syq}(Q, S)$.

ii) $Q;\mathrm{syq}(Q, S) = Q;\mathbb{T}_{B,C} \sqcap S$.

iii) $S \sqsubseteq \mathrm{syq}(Q^{\smile}, \mathrm{syq}(Q, S))$.

Proof:

i) By Lemma 4.8 iii) $Q^{\smile};S \sqsubseteq Q\backslash S$ and $Q^{\smile};S \sqsubseteq Q^{\smile}/S^{\smile}$ since Q is injective and $S^{\smile}$ is univalent. Thus we have $Q^{\smile};S \sqsubseteq \mathrm{syq}(Q, S)$.

ii) It is trivial that $Q;\mathrm{syq}(Q, S) \sqsubseteq Q;\mathbb{T}_{B,C} \sqcap S$. Conversely it holds that $Q;\mathbb{T}_{B,C} \sqcap S \sqsubseteq Q;(\mathbb{T}_{B,C} \sqcap Q^{\smile};S) = Q;Q^{\smile};S \sqsubseteq Q;\mathrm{syq}(Q, S)$ by i).

iii) By i) $Q^{\smile};S \sqsubseteq \mathrm{syq}(Q, S) \iff S \sqsubseteq Q^{\smile}\backslash\mathrm{syq}(Q, S)$. And

$$S;\mathrm{syq}(Q, S)^{\smile} = S;\mathrm{syq}(S, Q) \sqsubseteq Q \iff S \sqsubseteq Q/\mathrm{syq}(Q, S)^{\smile}$$

by Lemma 5.2.i). Therefore we have $S \sqsubseteq \mathrm{syq}(Q^{\smile}, \mathrm{syq}(Q, S))$. □

Lemma 6.8 [←21] Let $Q : A \leftrightarrow B$ and $R : A \leftrightarrow C$ be two relations, then totality of $\mathrm{syq}(Q, R)$ is equivalent to reflexivity of $(Q^{\smile}/R^{\smile});(R^{\smile}/Q^{\smile})$.

Proof: Assuming totality of $\mathrm{syq}(Q, R)$, we obtain (where $\mathbb{I} = \mathbb{I}_B$ throughout):

$$\begin{array}{rll}
\mathbb{I} & \sqsubseteq \ \mathrm{syq}(Q, R);\mathrm{syq}(Q, R)^{\smile} & \mathrm{syq}(Q, R) \text{ total} \\
& = \ \mathrm{syq}(Q, R);\mathrm{syq}(R, Q) & \text{Lemma 5.2.i)} \\
& = \ (Q\backslash R \sqcap Q^{\smile}/R^{\smile});(R\backslash Q \sqcap R^{\smile}/Q^{\smile}) & \text{Theorem 6.1} \\
& \sqsubseteq \ (Q^{\smile}/R^{\smile});(R^{\smile}/Q^{\smile}) \ . & \forall U, V \ : \ U \sqcap V \sqsubseteq U
\end{array}$$

Conversely, $\mathbb{I} \sqsubseteq (Q^{\smile}/R^{\smile});(R^{\smile}/Q^{\smile})$ is equivalent to $\mathbb{I} = (Q^{\smile}/R^{\smile});(R^{\smile}/Q^{\smile}) \sqcap \mathbb{I}$ and we may calculate:

$$\begin{array}{rll}
\mathbb{I} & = \ (Q^{\smile}/R^{\smile});(R^{\smile}/Q^{\smile}) \sqcap \mathbb{I} & (Q^{\smile}/R^{\smile});(R^{\smile}/Q^{\smile}) \text{ reflexive} \\
& \sqsubseteq \ (Q^{\smile}/R^{\smile} \sqcap \mathbb{I};(R^{\smile}/Q^{\smile})^{\smile});(R^{\smile}/Q^{\smile} \sqcap (Q^{\smile}/R^{\smile})^{\smile};\mathbb{I}) & \text{Dedekind} \\
& = \ (Q^{\smile}/R^{\smile} \sqcap Q\backslash R);(R^{\smile}/Q^{\smile} \sqcap R\backslash Q) & \text{Corr. 4.2} \\
& = \ \mathrm{syq}(Q, R);\mathrm{syq}(Q, R)^{\smile} \ . & \text{Theorem 6.1}
\end{array}$$

□

Lemma 6.9 [←23] Let $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ be given relations. Then it holds that

$$\mathrm{syq}(Q, S) \sqsubseteq \mathrm{syq}((Q\backslash S)^{\smile}, (S\backslash S)^{\smile}) \ .$$

Proof: From Lemma 4.9, it holds that

$$\begin{array}{rcl} \mathrm{syq}(Q, S) & = & Q\backslash S \sqcap Q^{\smile}/S^{\smile} \\ & \sqsubseteq & (S^{\smile}/Q^{\smile})\backslash(S^{\smile}/S^{\smile}) \sqcap (Q\backslash S)/(S\backslash S) \\ & = & \mathrm{syq}(S^{\smile}/Q^{\smile}, S^{\smile}/S^{\smile}) \\ & = & \mathrm{syq}((Q\backslash S)^{\smile}, (S\backslash S)^{\smile}) \ . \end{array}$$ □

7 Symmetric Quotients, Vectors, and Points

Next we show the relationship between symmetric quotients and vector relations.

Definition 7.1 A relation r with $r = \mathbb{T} ; r$ is called a **vector**, and a relation r with $r = r ; \mathbb{T}$ is called a **covector**.[2] A nonempty and univalent vector is called a **point**. □

For concrete binary relations between sets, vectors $r : A \leftrightarrow B$ can be considered as descriptions of subsets of B, and a point $p : A \leftrightarrow B$ corresponds to an element of B.

In heterogeneous relation algebras, the Tarski rule lets any nonempty vector be total, and the fact that $\mathbb{I} \neq \perp\!\!\!\perp$ lets total relations be nonempty. Therefore, in heterogeneous relation algebras, a point may equivalently be defined as a vector that is a mapping.

The following vector-related laws are elementary for practical proofs:

Proposition 7.2 [←17] [SS93, 2.4.2] For all relations $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$, $S : D \leftrightarrow C$ and $U : A \leftrightarrow E$ the following holds:

$$\begin{array}{rcl} Q ; R \sqcap \mathbb{T} ; S & = & Q ;(R \sqcap \mathbb{T} ; S) \\ Q ; R \sqcap U ; \mathbb{T} & = & (Q \sqcap U ; \mathbb{T}) ; R \\ (Q \sqcap \mathbb{T} ; S) ; R & = & Q ;(R \sqcap S^{\smile} ; \mathbb{T}) \end{array}$$ □

Restriction of a relational product by intersection with a vector is therefore equal to the product of the two relations where just the second one is intersected with the vector; that vector has of course be adapted to its new type by multiplication with a universal relation.

Certain residuals are always vectors:

Proposition 7.3 In Dedekind categories, for $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$, and $S : A \leftrightarrow C$, we have:

i) [←16] $S/\mathbb{T}_{B,C}$ is a covector and $\mathbb{T}_{A,B}\backslash S$ is a vector [Ohk98].

ii) [←17] $\perp\!\!\!\perp_{A,C}/R$ is a vector and $Q\backslash\perp\!\!\!\perp_{A,C}$ is a covector.

---

[2] In [SS93], our covectors are called vectors.

Proof:

i) By monotonicity of the composition we have $S/\mathbb{T}_{B,C} \sqsubseteq (S/\mathbb{T}_{B,C});\mathbb{T}_{B,B}$. Conversely it holds that $(S/\mathbb{T}_{B,C});\mathbb{T}_{B,B};\mathbb{T}_{B,C} = (S/\mathbb{T}_{B,C});\mathbb{T}_{B,C} \sqsubseteq S$ by Prop. 3.3. Thus we have $(S/\mathbb{T}_{B,C});\mathbb{T}_{B,B} \sqsubseteq S/\mathbb{T}_{B,C}$.

ii) It is trivial that $\perp\!\!\!\perp_{A,C}/R \sqsubseteq \mathbb{T}_{A,A};(\perp\!\!\!\perp_{A,C}/R)$. And it holds that $\mathbb{T}_{A,A};(\perp\!\!\!\perp_{A,C}/R);R \sqsubseteq \mathbb{T}_{A,A};\perp\!\!\!\perp_{A,C} = \perp\!\!\!\perp_{A,C}$. Then we have $\mathbb{T}_{A,A};(\perp\!\!\!\perp_{A,C}/R) \sqsubseteq \perp\!\!\!\perp_{A,C}/R$. □

Lemma 7.4 [←17] If in a Dedekind category $W : A \leftrightarrow B$ is a vector and $R : C \leftrightarrow B$ and $S : A \leftrightarrow C$ are arbitrary relations, then W/R and $W^{\smile}\backslash S$ are vectors, too.

Proof: With Lemma 4.5.i) we have $\mathbb{T}_{A,A};(W/R) \sqsubseteq (\mathbb{T}_{A,A};W)/R = W/R$. For the second statement we calculate:

$$\begin{aligned} W^{\smile}\backslash S \sqsubseteq W^{\smile}\backslash S &\iff W^{\smile};(W^{\smile}\backslash S) \sqsubseteq S \\ &\iff W^{\smile};\mathbb{T}_{A,A};(W^{\smile}\backslash S) \sqsubseteq S \\ &\iff \mathbb{T}_{A,A};(W^{\smile}\backslash S) \sqsubseteq W^{\smile}\backslash S\ . \end{aligned}$$ □

Proposition 7.5 [←22] Let both $Q : A \leftrightarrow B$ and $S : A \leftrightarrow C$ be given relations. Then the following holds:

i) $\operatorname{syq}(Q, \mathbb{T}_{A,A})$ is a covector and $\operatorname{syq}(\mathbb{T}_{A,A}, Q)$ is a vector.

ii) Let a Dedekind category be uniform. Then $\operatorname{syq}(Q, S) = \mathbb{T}_{B,C}$ if and only if Q and S are covectors and $Q;\mathbb{T}_{B,D} = S;\mathbb{T}_{C,D}$ holds.

iii) $\operatorname{syq}(Q, Q) = \mathbb{T}_{B,B}$ if and only if Q is a covector.

iv) $\operatorname{syq}(\perp\!\!\!\perp_{A,B}, S)$ is a vector.

Proof:

i) Since $\mathbb{T}_{A,A}$ is an equivalence relation, it holds that $\operatorname{syq}(\mathbb{T}_{A,A}, \mathbb{T}_{A,A}) = \mathbb{T}_{A,A}$, indeed $\operatorname{syq}(\mathbb{T}_{A,A}, \mathbb{T}_{A,A})$ is surjective. Thus we have

$$\operatorname{syq}(Q, \mathbb{T}_{A,A});\mathbb{T}_{A,A} = \operatorname{syq}(Q, \mathbb{T}_{A,A});\operatorname{syq}(\mathbb{T}_{A,A}, \mathbb{T}_{A,A}) = \operatorname{syq}(Q, \mathbb{T}_{A,A})$$

by Prop. 6.3 ii).

Alternatively, we can use residual properties; we first use Prop. 4.7.iii)to obtain:

$$\operatorname{syq}(Q, \mathbb{T}_{A,A}) = Q\backslash\mathbb{T}_{A,A} \sqcap Q^{\smile}/\mathbb{T}_{A,A} = \mathbb{T}_{B,A} \sqcap Q^{\smile}/\mathbb{T}_{A,A} = Q^{\smile}/\mathbb{T}_{A,A}\ ,$$

and with Prop. 7.3.i) we know that $Q^{\smile}/\mathbb{T}_{A,A}$ is a covector.

ii) Assume that $\operatorname{syq}(Q, S) = \mathbb{T}_{B,C}$. Then $\mathbb{T}_{B,C} \sqsubseteq Q\backslash S$ and $\mathbb{T}_{B,C} \sqsubseteq Q^{\smile}/S^{\smile}$ and via the definition of residuals

$$Q;\mathbb{T}_{B,C} \sqsubseteq S \quad\text{and}\quad \mathbb{T}_{B,C};S^{\smile} \sqsubseteq Q^{\smile}\ .$$

With this we may use uniformity to calculate

$$Q ; \mathbb{T}_{B,D} = Q ; \mathbb{T}_{B,C} ; \mathbb{T}_{C,D} \sqsubseteq S ; \mathbb{T}_{C,D} = S ; \mathbb{T}_{C,B} ; \mathbb{T}_{B,D} \sqsubseteq Q ; \mathbb{T}_{B,D} \ .$$

This implies

$$Q ; \mathbb{T}_{B,B} = S ; \mathbb{T}_{C,B} \sqsubseteq Q \quad \text{and} \quad S ; \mathbb{T}_{C,C} = Q ; \mathbb{T}_{B,C} \sqsubseteq S$$

since $Q ; \mathbb{T}_{B,D} = S ; \mathbb{T}_{C,D}$ for all objects D, so that Q and S are shown to be vectors.

Conversely we assume that $Q ; \mathbb{T}_{B,B} = Q$ and $S ; \mathbb{T}_{C,C} = S$, i.e., that Q and S are vectors, and $Q ; \mathbb{T}_{B,D} = S ; \mathbb{T}_{C,D}$. Then we have

$$S ; \mathbb{T}_{C,B} = Q ; \mathbb{T}_{B,B} = Q \ ,$$

yielding $\mathbb{T}_{B,C} \sqsubseteq Q^{\smile}/S^{\smile}$. Similarly we obtain $\mathbb{T}_{B,C} \sqsubseteq Q \backslash S$, so that $\operatorname{syq}(Q, S) = \mathbb{T}_{B,C}$ holds.

iii) follows directly from ii).

iv) follows from

$$\begin{array}{rcll}
\mathbb{T}_{B,B} ; \operatorname{syq}(\perp\!\!\!\perp_{A,B}, S) & = & \mathbb{T}_{B,B} ; (\perp\!\!\!\perp_{A,B} \backslash S \sqcap \perp\!\!\!\perp_{B,A}/S^{\smile}) & \\
& = & \mathbb{T}_{B,B} ; (\mathbb{T}_{B,C} \sqcap \perp\!\!\!\perp_{B,A}/S^{\smile}) & \text{Prop. 4.7.ii)} \\
& = & \mathbb{T}_{B,B} ; (\perp\!\!\!\perp_{B,A}/S^{\smile}) & \\
& = & \perp\!\!\!\perp_{B,A}/S^{\smile} & \text{Prop. 7.3.ii)} \\
& = & \mathbb{T}_{B,C} \sqcap \perp\!\!\!\perp_{B,A}/S^{\smile} & \\
& = & \perp\!\!\!\perp_{A,B} \backslash S \sqcap \perp\!\!\!\perp_{B,A}/S^{\smile} & \text{Prop. 4.7.ii)} \\
& = & \operatorname{syq}(\perp\!\!\!\perp_{A,B}, S) \ . & \qquad \square
\end{array}$$

Furthermore we may now show:

Lemma 7.6 [←24] If in a Dedekind category $W : A \leftrightarrow B$ is a covector and $S : A \leftrightarrow C$ is an arbitrary relation, then $\operatorname{syq}(W, S)$ is a vector and $\operatorname{syq}(S, W)$ is a covector.

Proof:
$$\begin{array}{rcll}
\operatorname{syq}(W, S) & = & (W \backslash S) \sqcap (W^{\smile}/S^{\smile}) & \text{Theorem 6.1} \\
& = & \mathbb{T}_{A,A} ; (W \backslash S) \sqcap \mathbb{T}_{A,A} ; (W^{\smile}/S^{\smile}) & \text{Lemma 7.4} \\
& = & \mathbb{T}_{A,A} ; ((W \backslash S) \sqcap \mathbb{T}_{A,A} ; (W^{\smile}/S^{\smile})) & \text{Prop. 7.2} \\
& = & \mathbb{T}_{A,A} ; ((W \backslash S) \sqcap (W^{\smile}/S^{\smile})) & \text{Lemma 7.4} \\
& = & \mathbb{T}_{A,A} ; \operatorname{syq}(W, S) & \text{Theorem 6.1} \qquad \square
\end{array}$$

8 Symmetric Quotients and Orderings

It turns out that residuals and symmetric quotients are extremely useful in the context of ordering. Many useful constructions, like upper bounds or suprema, allow a characterisation using these tools.

In [SS93], many such definitions an properties are provided, but all in the context of heterogeneous relation algebras.

In this section we reformulate these definitions and proofs only using the formalism of division allegories.

The results we thus obtain can also be used in a context like that of [Kaw98], which discusses "Lattices in Dedekind categories" starting from formalisations of the algebraic lattice definition and which later introduces the lattice ordering as a residual.

First of all, there are very nice properties of the symmetric quotient when applied to orderings (remember that the converse of an ordering is an ordering, too):

Lemma 8.1 If $E : A \leftrightarrow A$ is an ordering in some division allegory, then:

i) $\mathrm{syq}(E^{\smile}, E^{\smile}) \sqsubseteq \mathbb{I}$ and $\mathrm{syq}(E, E) \sqsubseteq \mathbb{I}$, and

ii) [←23] if $R : A \leftrightarrow B$ is some relation, then $\mathrm{syq}(R, E)$ and $\mathrm{syq}(R, E^{\smile})$ are univalent.

Proof:

i)
$$\begin{array}{rcll} \mathrm{syq}(E^{\smile}, E^{\smile}) & = & E^{\smile}\backslash E^{\smile} \sqcap E/E & \\ & \sqsubseteq & E^{\smile};(E^{\smile}\backslash E^{\smile}) \sqcap (E/E);E & \mathbb{I} \sqsubseteq E \\ & \sqsubseteq & E^{\smile} \sqcap E & \text{Prop. 4.3} \\ & \sqsubseteq & \mathbb{I} & \text{antisymmetry} \end{array}$$

The second statement has essentially the same proof.

ii) with Lemma 5.2.i), Corr. 6.4.i) and i) we have:

$$\mathrm{syq}(R, E)^{\smile};\mathrm{syq}(R, E) = \mathrm{syq}(E, R);\mathrm{syq}(E, R)^{\smile} \sqsubseteq \mathrm{syq}(E, E) \sqsubseteq \mathbb{I} \ . \qquad \square$$

We now turn to the basic ordering operations from [SS93]:

Definition 8.2 Let $Q : A \leftrightarrow B$ be an arbitrary relation and $E : B \leftrightarrow B$ an ordering. $\mathrm{ubd}_E(Q) = Q^{\smile}\backslash E$ and $\mathrm{lbd}_E(Q) = Q^{\smile}\backslash E^{\smile}$ are called the upper bound and lower bound of Q under a given ordering E, respectively. $\square$

Lemma 8.3 [←19] Let $Q : A \leftrightarrow B$ be an arbitrary relation and $E : B \leftrightarrow B$ an ordering. Then it holds that

$$\mathrm{ubd}_E(Q;E^{\smile}) = \mathrm{ubd}_E(Q) = \mathrm{ubd}_E(Q);E \quad \text{and} \quad \mathrm{lbd}_E(Q;E) = \mathrm{lbd}_E(Q) = \mathrm{lbd}_E(Q);E \ .$$

Proof: It is sufficient to prove the first equations. From transitivity of E, Prop. 4.4.ii) and Prop. 4.4.iii) we have $\mathrm{ubd}_E(Q) \sqsubseteq (E;Q^{\smile})\backslash(E;E) \sqsubseteq \mathrm{ubd}_E(Q;E^{\smile})$, and from reflexivity of E we have $\mathrm{ubd}_E(Q) \sqsubseteq \mathrm{ubd}_E(Q);E$. Conversely, using $Q^{\smile};X = \mathbb{I}_B;Q^{\smile};X \sqsubseteq E;Q^{\smile};X$ for arbitrary relation $X : A \leftrightarrow B$, we have,

$$\begin{array}{rcl} X \sqsubseteq (E;Q^{\smile})\backslash E = \mathrm{ubd}_E(Q;E^{\smile}) & \iff & E;Q^{\smile};X \sqsubseteq E \\ & \Longrightarrow & Q^{\smile};X \sqsubseteq E \\ & \iff & X \sqsubseteq Q^{\smile}\backslash E = \mathrm{ubd}_E(Q) \ . \end{array}$$

Also, for arbitrary relations $X : A \leftrightarrow B$, it holds that

$$\begin{array}{rcl} X \sqsubseteq (Q^\smile\backslash E);E = \mathrm{ubd}_E(Q);E & \Longrightarrow & Q^\smile;X \sqsubseteq Q^\smile;(Q^\smile\backslash E);E \sqsubseteq E;E \sqsubseteq E \\ & \Longleftrightarrow & X \sqsubseteq Q^\smile\backslash E = \mathrm{ubd}_E(Q) \end{array}$$

by Prop. 4.3 and transitivity of E. □

Definition 8.4 Let $Q : A \leftrightarrow B$ be an arbitrary relation and $E : B \leftrightarrow B$ an ordering. Then we call

- $\mathrm{gre}_E(Q) = Q \sqcap \mathrm{ubd}_E(Q)$ the greatest element.
- $\mathrm{lea}_E(Q) = Q \sqcap \mathrm{lbd}_E(Q)$ the least element.
- $\mathrm{lub}_E(Q) = \mathrm{lea}_E(\mathrm{ubd}_E(Q))$ the least upper bound.
- $\mathrm{glb}_E(Q) = \mathrm{gre}_E(\mathrm{lbd}_E(Q))$ the greatest lower bound. □

Lemma 8.5 [←19] Let $Q : A \leftrightarrow B$ be an arbitrary relation and $E : B \leftrightarrow B$ an ordering. Then it holds that

$$\mathrm{gre}_E(Q) = \mathrm{gre}_E(Q;E^\smile) \ .$$

Proof: "$\sqsubseteq$" is obvious from Lemma 8.5 and reflexivity of E. "$\sqsupseteq$" follows from

$$\begin{array}{rcll} \mathrm{gre}_E(Q;E^\smile) & = & Q;E^\smile \sqcap \mathrm{ubd}_E(Q;E^\smile) & \\ & = & Q;E^\smile \sqcap \mathrm{ubd}_E(Q) & \text{Lemma 8.3} \\ & \sqsubseteq & (Q \sqcap \mathrm{ubd}_E(Q);E)(E^\smile \sqcap Q^\smile;\mathrm{ubd}_E(Q)) & \text{Dedekind rule} \\ & \sqsubseteq & (Q \sqcap \mathrm{ubd}_E(Q))(E^\smile \sqcap E) & \text{Lemma 8.3 and Prop. 4.3} \\ & \sqsubseteq & \mathrm{gre}_E(Q) & \text{antisymmetry of } E \ . \end{array}$$
□

Proposition 8.6 Let $Q : A \leftrightarrow B$ be an arbitrary relation and $E : B \leftrightarrow B$ an ordering. Then it holds that

$$\mathrm{gre}_E(Q) = \mathrm{syq}(E;Q^\smile, E) \ \text{ and } \ \mathrm{lea}_E(Q) = \mathrm{syq}(E^\smile;Q^\smile, E^\smile) \ .$$

Proof: By the definitions,

$$\mathrm{syq}(E;Q^\smile, E) = (E;Q^\smile)\backslash E \sqcap (Q;E^\smile)/E^\smile \ \text{ and } \ \mathrm{gre}_E(Q;E^\smile) = Q;E^\smile \sqcap (E;Q^\smile)\backslash E \ .$$

So it is sufficient to prove $(Q;E^\smile)/E^\smile = Q;E^\smile$. "$\sqsubseteq$" follows from the transitivity of E via the residual property:

$$Q;E^\smile;E^\smile \sqsubseteq Q;E^\smile \Longleftrightarrow Q;E^\smile \sqsubseteq (Q;E^\smile)/E^\smile \ .$$

"$\sqsupseteq$"is obtained with the reflexivity of E and Prop. 4.3:

$$(Q;E^\smile)/E^\smile \sqsubseteq ((Q;E^\smile)/E^\smile);E^\smile \sqsubseteq Q;E^\smile \ .$$
□

Corollary 8.7 Let $Q : A \leftrightarrow B$ be an arbitrary relation and $E : B \leftrightarrow B$ an ordering. Then it holds that

$$\mathrm{lub}_E(Q) = \mathrm{syq}(\mathrm{ubd}_E(Q)^\smile, E^\smile) \quad \text{and} \quad \mathrm{glb}_E(Q) = \mathrm{syq}(\mathrm{lbd}_E(Q)^\smile, E) \ .$$

Proof: It follows from

$$\mathrm{lub}_E(Q) = \mathrm{lea}_E(\mathrm{ubd}_E(Q)) = \mathrm{syq}(E^\smile ; \mathrm{ubd}_E(Q)^\smile, E^\smile) = \mathrm{syq}(\mathrm{ubd}_E(Q)^\smile, E^\smile) \ . \qquad \square$$

9 Relational Specification of Sets

One of the most important applications of symmetric quotients is towards domain construction for programming language semantics, see [BSZ89; Zie91]. At the basis of these constructions is the simple direct power construction based on membership relations, which are conveniently characterised via symmetric quotients in heterogeneous relation algebras [BSZ89] as well as in division allegories [FS90, 2.41].

In this section we make the exact relation between these two approaches precise, and again transfer many proofs from the heterogeneous relation algebra setting to the more general division allegory and Dedekind category settings.

Definition 9.1 [←21–23] For two objects X and Z of a division allegory, a relation $\in : X \leftrightarrow Z$ is a *membership relation for* X, if:

i) $\mathrm{syq}(\in, \in) \sqsubseteq \mathbb{I}_Z$.

ii) $\mathrm{syq}(Q, \in)$ is total for all relations $Q : X \leftrightarrow A$.

Given such a membership relation $\in : X \leftrightarrow Z$, we define $\Omega_\in : Z \leftrightarrow Z$ as the *power ordering*:

iii) $\Omega = \in \backslash \in$.

The pair $(\in, \Omega)$ is called a *direct power*. $\square$

We still have to show that for every membership relation $\in : X \leftrightarrow Z$, the power ordering $\Omega = \in \backslash \in$ is indeed an ordering: Reflexivity and transitivity are obvious from Prop. 4.6.ii) and Prop. 4.4.i), respectively. Antisymmetry is satisfied since $\Omega \sqcap \Omega^\smile = \mathrm{syq}(\in, \in) \sqsubseteq \mathbb{I}_Z$ by i).

In concrete relation algebras with sets as objects and binary relations as morphisms, the archetypical membership relation for a set X is of course that between X and the powerset of X.

In heterogeneous relation algebras, totality of a relation R is usually expressed as "$R ; \mathbb{T} = \mathbb{T}$", where the universal relations need not be further specified by indices thanks to uniformity; therefore one frequently sees the condition ii) expressed as

"$\mathrm{syq}(Q^\smile, \in) ; \mathbb{T} = \mathbb{T}$ holds for all relations $Q : A \leftrightarrow X$".

Here, however, we rely on the allegory definition of totality, which is "$\mathbb{I} \sqsubseteq R;R^{\smile}$", and by Lemma 6.8, the totality of $\mathrm{syq}(Q^{\smile}, \in)$ is equivalent to the condition used in place of ii) by Freyd and Scedrov in [FS90, 2.41]:

$$\mathbb{I} \sqsubseteq (Q/{\in^{\smile}});({\in^{\smile}}/Q)$$

Therefore we have:

Theorem 9.2 A division allegory where membership relations exist for all objects is equivalent to a *power allegory* of Freyd and Scedrov [FS90, 2.41]. □

The above is a monomorphic definition of direct powers and indeed of membership relations:

Proposition 9.3 Let the two relations $\in_1 : X_1 \leftrightarrow Z_1$ and $\in_2 : X_2 \leftrightarrow Z_2$ be membership relations. If $\Phi : X_1 \leftrightarrow X_2$ is a bijective mapping, the following holds for $\Psi := \mathrm{syq}(\in_1, \Phi;\in_2) : Z_1 \leftrightarrow Z_2$:

i) Ψ is a bijective mapping.

ii) $\in_1;\Psi = \Phi;\in_2$.

Proof:

i) follows from

$$\begin{array}{rcll}
\Psi;\Psi^{\smile} & = & \mathrm{syq}(\in_1, \Phi;\in_2);\mathrm{syq}(\in_1, \Phi;\in_2)^{\smile} & \text{Def. } \Psi \\
& = & \mathrm{syq}(\in_1, \Phi;\in_2);\mathrm{syq}(\Phi;\in_2, \in_1) & \text{Lemma 5.2.i)} \\
& = & \mathrm{syq}(\in_1, \in_1) \sqcap \mathbb{T}_{Z_1,Z_2};\mathrm{syq}(\Phi;\in_2, \in_1) & \text{Prop. 6.3 i)} \\
& = & \mathbb{I}_{Z_1} \sqcap \mathbb{T}_{Z_1,Z_2};\mathrm{syq}(\in_2, \Phi^{\smile};\in_1) & \text{Def. 9.1 i), Prop. 5.3 iv)} \\
& = & \mathbb{I}_{Z_1} \sqcap \mathbb{T}_{Z_1,Z_1} & \text{Def. 9.1 ii)} \\
& = & \mathbb{I}_{Z_1} \ , &
\end{array}$$

$$\begin{array}{rcll}
\Psi^{\smile};\Psi & = & \mathrm{syq}(\in_1, \Phi;\in_2)^{\smile};\mathrm{syq}(\in_1, \Phi;\in_2) & \text{Def. } \Psi \\
& = & \mathrm{syq}(\Phi;\in_2, \in_1);\mathrm{syq}(\in_1, \Phi;\in_2) & \text{Lemma 5.2.i)} \\
& = & \mathrm{syq}(\in_2, \Phi^{\smile};\in_1);\mathrm{syq}(\Phi^{\smile};\in_1, \in_2) & \text{Prop. 5.3 iv)} \\
& = & \mathrm{syq}(\in_2, \in_2) \sqcap \mathbb{T}_{Z_2,Z_1};\mathrm{syq}(\Phi^{\smile};\in_1, \in_2) & \text{Prop. 6.3 i)} \\
& = & \mathbb{I}_{Z_2} \sqcap \mathbb{T}_{Z_2,Z_1};\mathrm{syq}(\Phi^{\smile};\in_1, \in_2) & \text{Def. 9.1 i), Prop. 5.3 iv)} \\
& = & \mathbb{I}_{Z_2} \sqcap \mathbb{T}_{Z_2,Z_2} & \text{Def. 9.1 ii)} \\
& = & \mathbb{I}_{Z_2} \ . &
\end{array}$$

ii) It holds that

$$\in_1;\Psi = \in_1;\mathrm{syq}(\in_1, \Phi;\in_2) = \Phi;\in_2 \sqcap \mathbb{T}_{X_1,Z_1};\mathrm{syq}(\in_1, \Phi;\in_2)$$

by the definition of Ψ and Prop. 6.2 i). From Def. 9.1 ii), $\mathrm{syq}(\in_1, \Phi;\in_2)^{\smile}$ is total. Thus we have $\mathbb{T}_{X_1,Z_2} = \mathbb{T}_{X_1,Z_1};\mathrm{syq}(\in_1, \Phi;\in_2)$ by Prop. A.2 iii). Therefore $\in_1;\Psi = \Phi;\in_2$ holds. □

While in the application to concrete relations Def. 9.1.ii) ensures that there is *at least one* powerset element for every set, we may observe that together with Def. 9.1.i) there is always *exactly one* such powerset element:

Lemma 9.4 If $\in : X \leftrightarrow Z$ is membership relation, then $\mathrm{syq}(Q, \in)$ is univalent for all $Q : X \leftrightarrow A$.

Proof: With Corr. 6.4.i) we have: $\mathrm{syq}(Q, \in)^{\smile};\mathrm{syq}(Q, \in) \sqsubseteq \mathrm{syq}(\in, \in) \sqsubseteq \mathbb{I}$ □

Definition 9.5 Let $\in : X \leftrightarrow Z$ be a membership relations. Then we define the following relations.

- Empty Set : $\emptyset : Y \leftrightarrow Z$ with $\emptyset = \mathrm{syq}(\perp\!\!\!\perp_{X,Y}, \in)$.
- Singleton : $\mathcal{S} : X \leftrightarrow Z$ with $\mathcal{S} = \mathrm{syq}(\mathbb{I}_X, \in)$.
- Inclusion : $\Omega : Z \leftrightarrow Z$ with $\in\backslash\in$. □

Note that $\emptyset = \perp\!\!\!\perp_{Y,X}/\in^{\smile}$by Prop. 4.7.ii).

Lemma 9.6 For the "empty set" $\emptyset := \mathrm{syq}(\perp\!\!\!\perp_{X,Y}, \in) : Y \leftrightarrow Z$, the following holds:

i) If $\mathbb{I}_Y \neq \perp\!\!\!\perp_{Y,Y}$, then $\emptyset$ is a point.

ii) $\emptyset;\in^{\smile} = \perp\!\!\!\perp_{Y,X}$.

Proof:

i) $\emptyset = \mathrm{syq}(\perp\!\!\!\perp, \in)$ is total by the definition of $\in$, a vector with Prop. 7.5 iv), and univalent wit Lemma 9.4. And it holds that

$$\begin{array}{rcll} \emptyset^{\smile};\emptyset & = & \mathrm{syq}(\perp\!\!\!\perp_{X,Y}, \in)^{\smile};\mathrm{syq}(\perp\!\!\!\perp_{X,Y}, \in) & \text{Def. } \emptyset \\ & \sqsubseteq & \mathrm{syq}(\in, \in) & \text{Corr. 6.4.i)} \\ & \sqsubseteq & \mathbb{I}_Z & \text{Def. 9.1 i)} \ . \end{array}$$

Next, assume that $\emptyset = \perp\!\!\!\perp_{Y,Z}$. Then, by Def. 9.1 ii), we have

$$\mathbb{I}_Y \sqsubseteq \mathrm{syq}(\perp\!\!\!\perp_{X,Y}, \in);\mathrm{syq}(\in, \perp\!\!\!\perp_{X,Y}) = \emptyset;\emptyset^{\smile} = \perp\!\!\!\perp_{Y,Y} \ .$$

This is contradiction to $\mathbb{I}_Y \neq \perp\!\!\!\perp_{Y,Y}$. So $\emptyset \neq \perp\!\!\!\perp_{Y,Z}$. Therefore $\emptyset$ is a point.

ii) follows from $\emptyset;\in^{\smile} = (\perp\!\!\!\perp_{Y,X}/\in^{\smile});\in^{\smile} \sqsubseteq \perp\!\!\!\perp_{Y,X}$. □

Lemma 9.7 [←24] For the singleton creator $\mathcal{S} := \mathrm{syq}(\mathbb{I}_X, \in) : X \leftrightarrow Z$, the following holds.

i) $\mathcal{S}$ is injective, total and univalent.

ii) $\mathcal{S};\in^{\smile} = \mathbb{I}_X$.

iii) $\mathcal{S};\Omega = \in$.

Proof:

i) The injectivity follows from

$$\begin{array}{rcll} \mathcal{S};\mathcal{S}^{\smile} & = & \mathrm{syq}(\mathbb{I}_X, \in);\mathrm{syq}(\mathbb{I}_X, \in)^{\smile} & \text{Def. } \mathcal{S} \\ & \sqsubseteq & \mathrm{syq}(\mathbb{I}_X, \mathbb{I}_X) & \text{Corr. 6.4.i)} \\ & = & \mathbb{I}_X & \text{Prop. 4.4.iii)} \ . \end{array}$$

By Def. 9.1 ii), $\mathcal{S}$ is total. The univalence follows from

$$\begin{array}{rcll} \mathcal{S}^{\smile};\mathcal{S} & = & \mathrm{syq}(\mathbb{I}_X, \in)^{\smile};\mathrm{syq}(\mathbb{I}_X, \in) & \text{Def. } \mathcal{S} \\ & \sqsubseteq & \mathrm{syq}(\in, \in) & \text{Corr. 6.4.i)} \\ & \sqsubseteq & \mathbb{I}_Z & \text{Def. 9.1 i)} \ . \end{array}$$

ii) By i), $\mathcal{S}$ is total. So $\mathcal{S}^{\smile}$ is surjective. And it holds that

$$(\mathcal{S};\in^{\smile})^{\smile} = \in;\mathcal{S}^{\smile} = \in;\mathrm{syq}(\mathbb{I}_X, \in)^{\smile} = \in;\mathrm{syq}(\in, \mathbb{I}_X) \ .$$

Thus we have $(\mathcal{S};\in^{\smile})^{\smile} = \mathbb{I}_X$ by Prop. 6.2 ii).

iii) Since $\mathcal{S}$ is a mapping, it holds that $\mathcal{S};\Omega = \mathcal{S};(\in\backslash\in) = (\in;\mathcal{S}^{\smile})\backslash\in = \mathbb{I}_X\backslash\in = \in$ by Lemma 4.8 i), ii) and Prop. 4.4.iii). □

Proposition 9.8 For the inclusion relation $\Omega = \in\backslash\in : Z \leftrightarrow Z$, and given relations $Q : A \leftrightarrow Z$ and $R : B \leftrightarrow X$, the following holds:

i) $\mathrm{lub}_\Omega(Q) = \mathrm{syq}(\in;Q^{\smile}, \in)$.

ii) $\mathrm{glb}_\Omega(Q) = \mathrm{syq}(\mathrm{lbd}_\Omega(Q), \Omega)$.

iii) $\mathrm{lub}_\Omega(R;\mathcal{S}) = \mathrm{syq}(R^{\smile}, \in)$.

iv) $\mathrm{lub}_\Omega(R/\in^{\smile}) = \mathrm{syq}(R^{\smile}, \in)$.

Proof:

i) It holds that

$$\begin{array}{rcll} \mathrm{syq}(\in;Q^{\smile}, \in) & \sqsubseteq & \mathrm{syq}(((\in;Q^{\smile})\backslash\in)^{\smile}, (\in\backslash\in)^{\smile}) & \text{Lemma 6.9} \\ & = & \mathrm{syq}(\in^{\smile}/(Q;\in^{\smile}), \in^{\smile}/\in^{\smile}) & \\ & = & \mathrm{syq}(\in^{\smile}/\in^{\smile}/Q, \in^{\smile}/\in^{\smile}) & \\ & = & \mathrm{syq}(\Omega^{\smile}/Q, \Omega^{\smile}) & \text{Def. } \Omega \\ & = & \mathrm{syq}((Q^{\smile}\backslash\Omega)^{\smile}, \Omega^{\smile}) & \\ & = & \mathrm{syq}(\mathrm{ubd}_\Omega(Q)^{\smile}, \Omega^{\smile}) & \text{Def. ubd} \\ & = & \mathrm{lub}_\Omega(Q) & \text{Corr. 8.7} \ . \end{array}$$

Also $\mathrm{syq}(\in;Q^{\smile}, \in)$ is total by Def. 9.1 ii) and $\mathrm{lub}_\Omega(Q)$ is univalent by Corr. 8.7 and Lemma 8.1.ii). Therefore, by Prop. A.3, we have $\mathrm{lub}_\Omega(Q) = \mathrm{syq}(\in;Q^{\smile}, \in)$.

ii) follows from Corr. 8.7.

iii) follows from

$$\begin{array}{rcll} \mathrm{lub}_\Omega(R\,;\mathcal{S}) & = & \mathrm{syq}(\in;\mathcal{S}^\smile;R^\smile,\in) & \text{i)} \\ & = & \mathrm{syq}(\mathbb{I}_X;R^\smile,\in) & \text{Lemma 9.7 ii)} \\ & = & \mathrm{syq}(\mathbb{I}_X;R^\smile,\in)\ . & \end{array}$$

iv) It holds that $\mathrm{lub}_\Omega(R/\in^\smile) = \mathrm{syq}(\in;(\in\backslash R^\smile),\in)$ by i). Now we show $\in;(\in\backslash R^\smile) = R^\smile$. It is trivial that $\in;(\in\backslash R^\smile) \sqsubseteq R^\smile$ by Prop. 4.3. Conversely it holds $R^\smile = \in;\mathcal{S}^\smile;R^\smile \sqsubseteq \in;(\in\backslash R^\smile)$ since $\in;\mathcal{S}^\smile = \mathbb{I}_X$ and

$$\in;\mathcal{S}^\smile;R^\smile \sqsubseteq R^\smile \iff \mathcal{S}^\smile;R^\smile \sqsubseteq \in\backslash R^\smile$$

hold by Lemma 9.7. □

Definition 9.9 [←24] Given a membership relation $\in : X \leftrightarrow Y$, every vector $t : A \leftrightarrow X$ can be assigned a corresponding point $e_t : A \leftrightarrow Y$ with $e_t = \mathrm{syq}(t^\smile,\in)$. Conversely, every point $e : A \leftrightarrow Y$ has a corresponding vector $t_e : A \leftrightarrow X$ with $t_e = e\,;\in^\smile$. □

Lemma 9.10 Given a membership relation $\in : X \leftrightarrow Y$, the conversions of Def. 9.9 are well-defined if $\mathbb{I}_X \neq \perp\!\!\!\perp_{X,X}$.

Proof: With Lemma 7.6, $e_t = \mathrm{syq}(t^\smile,\in)$ is a vector; its univalence follows from Lemma 9.4, and the definition of $\in$ gives its totality, which in turn implies nonemptyness if $\mathbb{I}_X \neq \perp\!\!\!\perp_{X,X}$, so that e_t is a point.

If e is a point, then it is a vector and $t_e = e\,;\in^\smile$ is of course a vector, too. □

Lemma 9.11 The correspondences between vectors and points defined in Def. 9.9 are isotone and mutually inverse:

i) $t_{e_t} = t$.

ii) If e is total, $e_{t_e} = e$.

iii) If e_2 is total, $e_1 \sqsubseteq e_2\,;\Omega^\smile \iff t_{e_1} \sqsubseteq t_{e_2}$.

iv) $t_1 \sqsubseteq t_2 \iff e_{t_1} \sqsubseteq e_{t_2}\,;\Omega^\smile$.

Proof:

i) Since $\mathrm{syq}(t^\smile,\in)$ is total, $\mathrm{syq}(t^\smile,\in)^\smile$ is surjective. Then we have:

$$t_{e_t} = e_t\,;\in^\smile = \mathrm{syq}(t^\smile,\in)\,;\in^\smile = (\in;\mathrm{syq}(\in,t^\smile))^\smile = t$$

by Prop. 6.2 ii).

ii) It holds that $e_{t_e} = \mathrm{syq}(t_e^\smile,\in) = \mathrm{syq}(\in;e^\smile,\in) = (\in;e^\smile)\backslash\in \sqcap (e\,;\in^\smile)/\in^\smile$. By Prop. 4.3 it is trivial that $e \sqsubseteq (e\,;\in^\smile)/\in^\smile$. And by univalence of e, it holds that $\in;e^\smile;e \sqsubseteq \in \iff e \sqsubseteq (e^\smile;e)\backslash\in$. Thus we have $e \sqsubseteq e_{t_e}$. Also $e_{t_e} = \mathrm{syq}(\in;e^\smile,\in)$ is univalent and e is total. Therefore we have $e = e_{t_e}$ by Prop. A.3.

iii) Assume that $e_1 \sqsubseteq e_2 ; \Omega^{\smile}$. Then we have

$$t_{e_1} = e_1 ; \in^{\smile} \sqsubseteq e_2 ; \Omega^{\smile} ; \in^{\smile} = e_2 ;(\in^{\smile}/\in^{\smile}) ; \in^{\smile} \sqsubseteq e_2 ; \in^{\smile} = t_{e_2}$$

by Prop. 4.3. Conversely, assume that $t_{e_1} \sqsubseteq t_{e_2}$ and e_2 is total. Then we have

$$\begin{array}{rcl} e_1 & \sqsubseteq & e_1 ; \Omega^{\smile} = e_1 ;(\in^{\smile}/\in^{\smile}) \sqsubseteq (e_1 ; \in^{\smile})/\in^{\smile} \\ & = & t_{e_1}/\in^{\smile} \sqsubseteq t_{e_2}/\in^{\smile} = (e_2 ; \in^{\smile})/\in^{\smile} = e_2 ;(\in^{\smile}/\in^{\smile}) = e_2 ; \Omega^{\smile} \end{array}$$

by Lemma 4.5.

iv) By i) it holds that

$$\begin{array}{rcl} t_1 \sqsubseteq t_2 & \iff & t_{e_{t_1}} \sqsubseteq t_{e_{t_2}} \\ & \iff & e_{t_1} ; \in^{\smile} \sqsubseteq e_{t_2} ; \in^{\smile} \\ & \iff & e_{t_1} \sqsubseteq (e_{t_2} ; \in^{\smile})/\in^{\smile} \ . \end{array}$$

Since e_{t_2} is total by the definition of $\in$, it holds that $(e_{t_2} ; \in^{\smile})/\in^{\smile} = e_{t_2} ;(\in^{\smile}/\in^{\smile}) = e_{t_2} ; \Omega^{\smile}$ by Lemma 4.5.ii). Thus we have $e_{t_1} \sqsubseteq e_{t_2} ; \Omega^{\smile}$. Conversely, assume that $e_{t_1} \sqsubseteq e_{t_2} ; \Omega^{\smile}$. Then we have $e_{t_1} \sqsubseteq e_{t_2} ;(\in^{\smile}/\in^{\smile}) = (e_{t_2} ; \in^{\smile})/\in^{\smile}$ by Lemma 4.5.ii). Thus, by i), it holds that

$$t_1 = t_{e_{t_1}} = e_{t_1} ; \in^{\smile} \sqsubseteq e_{t_2} ; \in^{\smile} = t_{e_{t_2}} = t_2$$

since $e_{t_1} \sqsubseteq (e_{t_2} ; \in^{\smile})/\in^{\smile} \iff e_{t_1} ; \in^{\smile} \sqsubseteq e_{t_2} ; \in^{\smile}$. □

10 Outlook and Conclusion

The importance of symmetric quotients is based mostly on the fact that they allow to formalise comparatively complex concepts using relatively simple algebraic properties.

Since most material about such applications of symmetric quotients has been written against a background of heterogeneous relation algebras, it is not always immediately clear which part of this is also usable in more general contexts like Dedekind categories, division allegories, or even just distributive allegories.

In this paper we have strived to extend the applicability of symmetric quotients by investigating their axiomatisation and their properties in these more general contexts.

We think that especially the introduction of symmetric quotients in distributive allegories and their properties therein deserve some further investigation.

Also the question how symmetric quotients interact with negation in weaker axiomatisations or even with pseudo-complements seems to be a very interesting one.

On the whole, we hope that this study has opened up the use of symmetric quotients and of their existing applications to applications in different areas where weaker axiomatisations of relation-like structures are essential.

We express our heartfelt thanks to Prof. Gunther Schmidt and Prof. Yasuo Kawahara for making this cooperation possible.

We are also grateful to Michael Winter for his comments on a draft version of this report.

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A Basic Properties of Relations

We here list a few basic definitions and properties of relations.

Definition A.1 If $Q : A \leftrightarrow B$ is a relation, we call

- Q *total* if and only if $\mathbb{I}_A \sqsubseteq Q ; Q^{\smile}$,
- Q *univalent* if and only if $Q^{\smile} ; Q \sqsubseteq \mathbb{I}_B$,
- Q *surjective* if and only if $\mathbb{I}_B \sqsubseteq Q^{\smile} ; Q$,
- Q *injective* if and only if $Q ; Q^{\smile} \sqsubseteq \mathbb{I}_A$,
- Q *mapping* if and only if Q is total and univalent. □

Proposition A.2 [←12, 13, 21] Let $Q : A \leftrightarrow B$, $R, R' : B \leftrightarrow C$, $S : A \leftrightarrow C$ and $T : C \leftrightarrow D$ be relations. Then the following holds:

i) If Q is univalent, then $Q ;(R \sqcap R') = Q ; R \sqcap Q ; R'$.

ii) If T is injective, then $(R \sqcap R') ; T = R ; T \sqcap R' ; T$.

iii) Q is total if and only if $\mathbb{T}_{A,C} = Q ; \mathbb{T}_{B,C}$.

iv) R is surjective if and only if $\mathbb{T}_{A,C} = \mathbb{T}_{A,B} ; R$.

v) If Q is a mapping, then $S \sqsubseteq Q ; R \iff Q^{\smile} ; S \sqsubseteq R$. □

Proposition A.3 [←23, 24] Let a relation $Q : A \leftrightarrow B$ be total and a relation $Q' : A \leftrightarrow B$ be univalent. Then if $Q \sqsubseteq Q'$ holds, $Q = Q'$. □

For homogeneous relations, the following properties can apply:

Definition A.4 If $P : A \leftrightarrow A$ is a relation, we call

- P *reflexive* if and only if $\mathbb{I}_A \sqsubseteq P$.
- P *transitive* if and only if $P ; P \sqsubseteq P$.
- P *symmetric* if and only if $P^{\smile} \sqsubseteq P$.
- P *antisymmetric* if and only if $P \sqcap P^{\smile} \sqsubseteq \mathbb{I}_A$.
- P an *equivalence relation* if and only if P is reflexive, transitive and symmetric.
- P a *preorder* if and only if P is reflexive and transitive.
- P an *ordering* if and only if P is reflexive, transitive and antisymmetric. □

Note that a relation $P : A \leftrightarrow A$ is symmetric if and only if $P = P^{\smile}$.

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