On Binary Relations

Hitoshi Furusawa (Faculty of Science, Kagoshima University)

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1 Introduction

We have already learnt about binary relations as basics of mathematics and computer science. Especially, equivalence relations and orderings appear frequently. However, it is harder to become familiar with binary relations than mappings having known and used since we were junior high school students. Don't you feel difficult to prove properties of very essential notions such as equivalence classes, supremum and infimum using notions of equivalence relations and orderings? In this lecture, we study some of basic properties of binary relations. The author hope that readers will feel much closer to binary relations through this lecture.

2 Basic Definitions and Properties

We start from defining binary relations.

Definition 2.1 *A* **(binary) relation** α *from a set* A *to a set* B*, written* $\alpha\colon A \rightharpoondown B$*, is a subset of Cartesian product* $A \times B$*.*

Thus, the set $\mathbf{Rel}(A, B)$ of relations from a set A to a set B is equal to the power set $\wp(A \times B)$ of $A \times B$.

0_{AB} and ∇_{AB} denote the empty relation and universal relation (or entire Cartesian product) $A \times B$, respectively.

For relations $\alpha, \beta\colon A \rightharpoondown B$, **inclusion** $\alpha \subseteq \beta$ is defined set theoretically, that is,

$$\alpha \subseteq \beta \overset{\mathsf{def}}{\iff} \forall (a, b) \in A \times B.((a, b) \in \alpha \Rightarrow (a, b) \in \beta) \enspace .$$

The empty relation 0_{AB} and the universal relation ∇_{AB} are the greatest and the least, respectively, in $\mathbf{Rel}(A, B)$ with respect to $\subseteq$. The **identity relation** $\mathsf{id}_A\colon A \rightharpoondown A$ on a set A is the set of diagonals of $A \times A$, that is,

$$\mathsf{id}_A \overset{\mathsf{def}}{=} \{(a, a) \in A \times A \mid a \in A\} \enspace .$$

1 denotes a fixed singleton set $\{*\}$. Then, $\mathsf{id}_1 = \nabla_{11}$ holds.

Definition 2.2 *For* $\alpha\colon A \rightharpoondown B$ *and* $\beta\colon B \rightharpoondown C$, **composition** $\alpha\beta\colon A \rightharpoondown C$ *of* α *followed by* β *is defined as follows.*

$$\alpha\beta \stackrel{\mathsf{def}}{=} \{(a,c) \in A \times C \mid \exists b \in B.((a,b) \in \alpha \text{ and } (b,c) \in \beta)\}$$

Then, we have the following proposition.

Proposition 2.3 *For* $\alpha, \alpha'\colon A \rightharpoondown B$, $\beta, \beta'\colon B \rightharpoondown C$, *and* $\gamma\colon C \rightharpoondown D$, *it holds that*

1. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$,
2. $\mathsf{id}_A\alpha = \alpha = \alpha\mathsf{id}_B$,
3. $0_{XA}\alpha = 0_{XB}$, $\alpha 0_{BY} = 0_{AY}$,
4. $\alpha \subseteq \alpha'$ *and* $\beta \subseteq \beta'$ *implies* $\alpha\beta \subseteq \alpha'\beta'$,
5. $B \neq \emptyset$ *implies* $\nabla_{AB}\nabla_{BC} = \nabla_{AC}$.

Definition 2.4 *For a family* $\{\alpha_\lambda\colon A \rightharpoondown B \mid \lambda \in \Lambda\}$ *of relations from* A *to* B, *the* **union relation** $\cup_{\lambda\in\Lambda}\alpha_\lambda$ *and the* **intersection relation** $\cap_{\lambda\in\Lambda}\alpha_\lambda$ *are the set theoretical union and intersection, respectively:*

$$\begin{aligned} \cup_{\lambda\in\Lambda}\alpha_\lambda &\stackrel{\mathsf{def}}{=} \{(a,b) \in A \times B \mid \exists\lambda \in \Lambda.((a,b) \in \alpha_\lambda)\} \ , \\ \cap_{\lambda\in\Lambda}\alpha_\lambda &\stackrel{\mathsf{def}}{=} \{(a,b) \in A \times B \mid \forall\lambda \in \Lambda.((a,b) \in \alpha_\lambda)\} \ . \end{aligned}$$

Then, we have the following propositions.

Proposition 2.5 *For* $\alpha, \beta_\lambda\colon A \rightharpoondown B$ $(\lambda \in \Lambda)$,

1. $\alpha \cap (\cup_{\lambda\in\Lambda}\beta_\lambda) = \cup_{\lambda\in\Lambda}(\alpha \cap \beta_\lambda)$,
2. $\alpha \cup (\cap_{\lambda\in\Lambda}\beta_\lambda) = \cap_{\lambda\in\Lambda}(\alpha \cup \beta_\lambda)$.

Proposition 2.6 *For* $\alpha\colon A \rightharpoondown B$, $\beta_\lambda\colon B \rightharpoondown C$ $(\lambda \in \Lambda)$, *and* $\gamma\colon C \rightharpoondown D$,

1. $\alpha(\cup_{\lambda\in\Lambda}\beta_\lambda) = \cup_{\lambda\in\Lambda}(\alpha\beta_\lambda)$, $(\cup_{\lambda\in\Lambda}\beta_\lambda)\gamma = \cup_{\lambda\in\Lambda}(\beta_\lambda\gamma)$,
2. $\alpha(\cap_{\lambda\in\Lambda}\beta_\lambda) \subseteq \cap_{\lambda\in\Lambda}(\alpha\beta_\lambda)$, $(\cap_{\lambda\in\Lambda}\beta_\lambda)\gamma \subseteq \cap_{\lambda\in\Lambda}(\beta_\lambda\gamma)$.

Proposition 2.6 says that the composition is distributive over the union but isn't so over the intersection. Let's consider counterexample.

Example 2.7 *Consider the case* $\Lambda = \{1,2\}$ *and* $A = \{a,b\}$. *Let* $\alpha, \beta_1, \beta_2\colon A \rightharpoondown A$ *be*

$$\alpha = \{(a,a),(a,b)\}, \quad \beta_1 = \{(a,a)\}, \quad \beta_2 = \{(b,a)\} \ .$$

Then,

$$\beta_1 \cap \beta_2 = 0_{AA}$$

and

$$\alpha\beta_1 = \{(a,a)\} = \alpha\beta_2$$

hold. Therefore

$$\begin{aligned}\alpha(\cap_{\lambda\in\Lambda}\beta_\lambda) = \alpha(\beta_1 \cap \beta_2) = \alpha 0_{AA} = 0_{AA} = \emptyset \\ \subsetneq \{(a,a)\} = \alpha\beta_1 \cap \alpha\beta_2 = \cap_{\lambda\in\Lambda}(\alpha\beta_\lambda)\ .\end{aligned}$$

Definition 2.8 *For $\alpha\colon A \rightharpoondown B$, the* **converse relation** *$\alpha^\sharp\colon B \rightharpoondown A$ is defined as follows.*

$$\alpha^\sharp \stackrel{\mathsf{def}}{=} \{(b,a) \in B \times A \mid (a,b) \in \alpha\}\ .$$

Proposition 2.9 *For $\alpha, \alpha', \alpha_\lambda\colon A \rightharpoondown B$ ($\lambda \in \Lambda$), and $\beta\colon B \rightharpoondown C$, the following holds.*

1. $\alpha^{\sharp\sharp} = \alpha$,
2. $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$,
3. $\alpha \subseteq \alpha'$ *implies* $\alpha^\sharp \subseteq {\alpha'}^\sharp$,
4. $(\cup_{\lambda\in\Lambda}\alpha_\lambda)^\sharp = \cup_{\lambda\in\Lambda}\alpha_\lambda^\sharp$,
5. $(\cap_{\lambda\in\Lambda}\alpha_\lambda)^\sharp = \cap_{\lambda\in\Lambda}\alpha_\lambda^\sharp$,
6. $0_{AB}^\sharp = 0_{BA},\ \nabla_{AB}^\sharp = \nabla_{BA},\ \mathsf{id}_A^\sharp = \mathsf{id}_A$.

The following properties are called **Dedekind formulae**.

Proposition 2.10 *For $\alpha\colon A \rightharpoondown B$, $\beta\colon B \rightharpoondown C$, and $\gamma\colon A \rightharpoondown C$, it holds that*

1. $\alpha\beta \cap \gamma \subseteq \alpha(\beta \cap \alpha^\sharp\gamma)$,
2. $\alpha\beta \cap \gamma \subseteq (\alpha \cap \beta\gamma^\sharp)\gamma$.

3 Mappings

Let's recall a notion of mappings.

Definition 3.1 *A mapping $\alpha\colon A \to B$ is a relation satisfying the following:*

For each $a \in A$, $\{b \in B \mid (a,b) \in \alpha\}$ is a singleton set.

Map(A,B) denotes the set of mappings from A to B. Clearly, ∇_{A1} is a unique mapping from A to 1, that is, **Map**$(A,1) = \{\nabla_{A1}\}$.

The requirement for mappings is equivalent to the following two statement.

Univalency $(a,b) \in \alpha$ and $(a,b') \in \alpha$ imply $b = b'$

Totality $\forall a \in A.\exists b \in B.(a,b) \in \alpha$

Using composition relation, converse relation, and inclusion between relations, the notions of univalency and totality may be translated into point-free style formulae.

Univalency $\alpha^\sharp\alpha \subseteq \mathsf{id}_B$

Totality $\mathsf{id}_A \subseteq \alpha\alpha^\sharp$

The following properties are proved in point-free style if we use the above formalisations of univalency and totality.

Proposition 3.2 *For mappings $f, h\colon A \to B$, $g\colon B \to C$ the following holds.*

1. *If f and g are mappings, then so is fg.*
2. *If $f \subseteq h$, then $f = h$.*

Proof. 1. fg is univalent since

$$(fg)^\sharp(fg) = g^\sharp f^\sharp fg \subseteq g^\sharp \mathsf{id}_B g = g^\sharp g \subseteq \mathsf{id}_C \ .$$

Also, fg is total since

$$\mathsf{id}_A \subseteq ff^\sharp = f\mathsf{id}_B f^\sharp \subseteq fgg^\sharp f^\sharp = (fg)(fg)^\sharp \ .$$

2. Suppose $f \subseteq h$, then $f^\sharp \subseteq h^\sharp$. So we have

$$h = \mathsf{id}_A h \subseteq ff^\sharp h \subseteq fh^\sharp h \subseteq f\mathsf{id}_B = f \ .$$

We only use totality of f and univalency of h in the proof of the second of Proposition 3.2. Thus, the next is immediate.

Corollary 3.3 *For total relation $\alpha\colon A \rightharpoonup B$ and univalent relation $\beta\colon A \rightharpoonup B$, if $\alpha \subseteq \beta$, then α and β are mappings satisfying $\alpha = \beta$.*

The next proposition shows that each relation may be decomposed to two mappings.

Proposition 3.4 *For a relation $\alpha\colon A \rightharpoonup B$, there exist mappings $f\colon R \to A$, $g\colon R \to B$ satisfying $\alpha = f^\sharp g$ and $ff^\sharp \cap gg^\sharp = \mathsf{id}_R$.*

Proof. Let $R = \alpha$, $((a,b),a') \in f \iff a = a'$, and $((a,b),b') \in g \iff b = b'$, then f and g are mappings satisfying

$$\begin{array}{ll} & (a,b) \in f^\sharp g \\ \iff & \exists (a',b') \in \alpha.(a,(a',b')) \in f^\sharp \text{ and } ((a',b'),b) \in g \\ \iff & \exists (a',b') \in \alpha.(a = a' \text{ and } b' = b) \\ \iff & (a,b) \in \alpha \ . \end{array}$$

So $\alpha = f^\sharp g$ holds. By the definition of f and g,

$$\begin{array}{ll} & ((a,b),(a',b')) \in ff^\sharp \cap gg^\sharp \\ \Longleftrightarrow & ((a,b),(a',b')) \in ff^\sharp \text{ and } ((a,b),(a',b')) \in gg^\sharp \\ \Longleftrightarrow & a = a' \text{ and } b = b' \\ \Longleftrightarrow & (a,b) = (a',b') \\ \Longleftrightarrow & ((a,b),(a',b')) \in \mathsf{id}_R \ . \end{array}$$

So $ff^\sharp \cap gg^\sharp = \mathsf{id}_R$ holds.

As the notion of mappings, notions of surjections, injections, and bijections may be formalised in point-free style using composition relations, converse relations, and inclusion between relations.

A mapping $f\colon A \to B$ is

surjective $\Leftrightarrow$ $\mathsf{id}_B \subseteq f^\sharp f$,

injective $\Leftrightarrow$ $ff^\sharp \subseteq \mathsf{id}_A$,

bijective $\Leftrightarrow$ $\mathsf{id}_B \subseteq f^\sharp f$ and $ff^\sharp \subseteq \mathsf{id}_A$.

If we use the above formalisation of bijections, the following property which must appear in each undergraduate text book of introduction to mathematics is shown by simple calculus.

Proposition 3.5 *A mapping $f\colon A \to B$ has the inverse function if and only if f is bijective.*

Proof. It is obvious that f has the inverse function if and only if $f^\sharp$ is a mapping. Suppose that f is bijective, then it holds that

$$(f^\sharp)^\sharp f^\sharp = ff^\sharp \subseteq \mathsf{id}_A$$

by the injectivity of f. So $f^\sharp$ is univalent. Also, since f is surjective, it holds that

$$\mathsf{id}_B \subseteq f^\sharp f = f^\sharp (f^\sharp)^\sharp \ .$$

So $f^\sharp$ is total. Therefore, $f^\sharp$ is a mapping. Conversely, we suppose that $f^\sharp$ is a mapping. Then, since $f^\sharp$ is univalent, it holds that

$$ff^\sharp = (f^\sharp)^\sharp f^\sharp \subseteq \mathsf{id}_A \ .$$

So f is injective. Since $f^\sharp$ is total, we have

$$\mathsf{id}_B \subseteq f^\sharp (f^\sharp)^\sharp = f^\sharp f \ .$$

So f is surjective.

Though it is possible to show that $f^\sharp$ is bijective if so is f similarly, we omit details.

A mapping is decomposed to a surjection and a injection. Usually, given $f\colon A \to B$, if we take the image $f(A) = \{b \in B \mid f(a) = b\}$ of A under f, essentially the same mapping $\hat{f}\colon A \to f(A)$ as f, i.e., $\hat{f}$ is a mapping defined by $\hat{f}(a) \stackrel{\mathsf{def}}{=} f(a)$, and the inclusion $i\colon f(A) \to B$, then clearly $\hat{f}$ is surjective, i is injective, and also $f = \hat{f}i$ holds.

In the usual construction, we may treat elements of sets without any inhibition. Here, we demonstrate the same construction in point-free style as far as possible.

Proposition 3.6 *For a mapping $f\colon A \to B$, there exists an injection $m\colon X \to B$ satisfying $m^\sharp m = f^\sharp f$.*

Proof. By Proposition 3.4, there exist $m\colon X \to B$ and $g\colon X \to 1$ satisfying $f^\sharp \nabla_{A1} = m^\sharp g$ and $mm^\sharp \cap gg^\sharp = \mathsf{id}_X$ for a relation $f^\sharp \nabla_{A1}$. Since ∇_{X1} is a unique mapping from X to 1, $g = \nabla_{X1}$ holds. So, it holds that

$$mm^\sharp = mm^\sharp \cap \nabla_{XX} = mm^\sharp \cap \nabla_{X1}\nabla_{X1}^\sharp = mm^\sharp \cap gg^\sharp = \mathsf{id}_X \ .$$

Therefore m is injective. Since $f^\sharp \nabla_{A1} = m^\sharp \nabla_{X1}$ holds,

$$f^\sharp \nabla_{A1}\nabla_{1A} = m^\sharp \nabla_{X1}\nabla_{1A} \text{ and } f^\sharp \nabla_{A1}\nabla_{1X} = m^\sharp \nabla_{X1}\nabla_{1X} \ ,$$

i.e.,

$$f^\sharp \nabla_{AA} = m^\sharp \nabla_{XA} \text{ and } f^\sharp \nabla_{AX} = m^\sharp \nabla_{XX} \ .$$

So it holds that

$$\begin{aligned} f^\sharp \nabla_{AA} f &= m^\sharp \nabla_{XA} f \\ &= m^\sharp \nabla_{XA} f^{\sharp\sharp} \\ &= m^\sharp (f^\sharp \nabla_{AX})^\sharp \\ &= m^\sharp (m^\sharp \nabla_{XX})^\sharp \\ &= m^\sharp \nabla_{XX} m \ . \end{aligned}$$

Thus, using Dedekind formula introduced in Proposition 2.10, we have.

$$\begin{aligned} f^\sharp f &= f^\sharp f \cap f^\sharp \nabla_{AA} f \\ &= f^\sharp f \cap m^\sharp \nabla_{XX} m \\ &\subseteq \mathsf{id}_B \cap m^\sharp \nabla_{XX} m \\ &\subseteq m^\sharp (m \cap \nabla_{XX} m) \\ &= m^\sharp m \end{aligned}$$

Similarly, it is possible to prove $m^\sharp m \subseteq f^\sharp f$.

The set X and injection $m\colon X \to B$ given in Proposition 3.6 corresponds to the set $f(A)$ and the inclusion $i\colon F(A) \to B$. The next proposition shows a construction of a surjection corresponds to $\hat{f}$.

Proposition 3.7 *For a mapping $f\colon A \to B$ and an injection $m\colon X \to B$, if $m^\sharp m = f^\sharp f$, then there exists a unique surjection $g\colon A \to X$ such that $f = gm$.*

Proof. Since $fm^\sharp\colon A \rightharpoondown X$ satisfies

$$\begin{array}{l} \mathsf{id}_A = \mathsf{id}_A\mathsf{id}_A \subseteq ff^\sharp ff^\sharp = fm^\sharp mf^\sharp = (fm^\sharp)(fm^\sharp)^\sharp \\ (fm^\sharp)^\sharp(fm^\sharp) = mf^\sharp fm^\sharp = mm^\sharp mm^\sharp = \mathsf{id}_X\mathsf{id}_X = \mathsf{id}_X \ , \end{array}$$

$fm^\sharp$ is a surjection. Also, $(fm^\sharp)m = ff^\sharp f$ holds. By totality and univalency of f, $f = \mathsf{id}_A f \subseteq ff^\sharp f$ and $ff^\sharp f \subseteq f\mathsf{id}_A = f$ hold. So we have $ff^\sharp f = f$. Therefore $f = (fm^\sharp)m$. Letting $g = fm^\sharp$, g is a surjection satisfying $f = gm$. Assume that $h\colon A \to X$ is a surjection which satisfies $f = hm$, $h = h\mathsf{id}_X = hmm^\sharp = fm^\sharp = g$ since m is injective.

Proposition 3.7 shows that $\hat{f}$ corresponds to $fm^\sharp$.

Theorem 3.8 *For a mapping $f\colon A \to B$, there exist a surjection $g\colon A \to X$ and injection $m\colon X \to B$ such that $f = gm$.*

4 Membership Relations

We often deal with a relation from A to B as a mapping from A to $\wp(B)$. Let see carefully the reason why we may do so.

Definition 4.1 *The* **membership relation** $\ni_A\colon \wp(A) \rightharpoondown A$ *of A is defined by*

$$\ni_A \stackrel{\mathsf{def}}{=} \{(S,a) \in \wp(A) \times A \mid a \in S\} \ .$$

Lemma 4.2 *For mappings $f, g\colon A \to \wp(B)$, if $f \ni_B = g \ni_B$, then $f = g$.*

Proof. For each $a \in A$, we show $f(a) = g(a)$.

$$\begin{array}{rcl} b \in f(a) & \iff & \exists S \subseteq B.((a,S) \in f \text{ and } b \in S) \\ & \iff & \exists S \subseteq B.((a,S) \in f \text{ and } (S,b) \in \ni_B) \\ & \iff & (a,b) \in f \ni_B \\ & \iff & (a,b) \in g \ni_B \\ & \iff & \exists S \subseteq B.((a,S) \in g \text{ and } (S,b) \in \ni_B) \\ & \iff & \exists S \subseteq B.((a,S) \in g \text{ and } b \in S) \\ & \iff & b \in g(a) \ . \end{array}$$

Lemma 4.3 *For a relation $\alpha\colon A \rightharpoondown B$, there exists a unique mapping $f\colon A \to \wp(B)$ such that $\alpha = f \ni_B$.*

Proof. Define $f\colon A \to \wp(B)$ by

$$(a,S) \in f \iff S = \{b \in B \mid (a,b) \in \alpha\} \ .$$

Then, it is obvious that such a set $S \subseteq B$ is uniquely determined for each $a \in A$. Next, we show $\alpha = f \ni_B$.

$$\begin{array}{rl} & (a,b) \in f \ni_B \\ \iff & \exists S \in \wp(B).((a,S) \in f \text{ and } (S,b) \in \ni_B) \\ \iff & \exists S \in \wp(B).((a,S) \in f \text{ and } b \in S) \\ \iff & S = \{b' \in B \mid (a,b') \in \alpha\} \text{ and } b \in S \\ \iff & (a,b) \in \alpha \ . \end{array}$$

Uniqueness of f is derived from Lemma 4.2.

Conversely, for a mapping $f\colon A \to \wp(B)$, we always have a relation $f \ni_B \colon A \rightharpoondown B$. Thus we have the following theorem.

Theorem 4.4 $\mathbf{Rel}(A, B) \cong \mathbf{Map}(A, \wp(B))$

5 Orderings, Equivalence Relations

Orderings and equivalence relations are examples appearing frequently in mathematics. Let's recall these notions.

Definition 5.1 *A relation $\rho\colon A \rightharpoondown A$ is called a* **(partial) ordering** *on A if it satisfies the following three conditions.*

reflexive law $\forall a \in A.((a, a) \in \rho)$

transitive law $(a, b) \in \rho$ *and* $(b, c) \in \rho \Rightarrow (a, c) \in \rho$

antisymmetric law $(a, b) \in \rho$ *and* $(b, a) \in \rho \Rightarrow a = b$

An ordering ρ on A which satisfies

linear law $\forall a, b \in A.((a, b) \in \rho$ *or* $(b, a) \in \rho)$

is called a **total (or linear) ordering**.

Definition 5.2 *A reflexive and transitive relation $\rho\colon A \rightharpoondown A$ satisfying*

symmetric law $(a, b) \in \rho \Rightarrow (b, a) \in \rho$

is called an **equivalence relation**.

Each "law" which appeared above can be formalised in point-free style, that is, a relation $\rho\colon A \rightharpoondown A$ satisfies

satisfies **reflexive law** $\iff \mathsf{id}_A \subseteq \rho$,

satisfies **transitive law** $\iff \rho\rho \subseteq \rho$,

satisfies **antisymmetric law** $\iff \rho \cap \rho^\sharp \subseteq \mathsf{id}_A$,

satisfies **linear law** $\iff \rho \cup \rho^\sharp = \nabla_{AA}$,

satisfies **symmetric law** $\iff \rho^\sharp \subseteq \rho$.

So, orderings and equivalence relations on A is formalised as follows.

ρ is a **ordering** on $A \iff \mathsf{id}_A \subseteq \rho$ and $\rho\rho \subseteq \rho$ and $\rho \cap \rho^\sharp \subseteq \mathsf{id}_A$.

ρ is a **total ordering**
$\iff \mathsf{id}_A \subseteq \rho$ and $\rho\rho \subseteq \rho$ and $\rho \cap \rho^\sharp \subseteq \mathsf{id}_A$ and $\rho \cup \rho^\sharp = \nabla_{AA}$.

ρ is an **equivalence relation** on $A \iff \mathsf{id}_A \subseteq \rho$ and $\rho\rho \subseteq \rho$ and $\rho^\sharp \subseteq \rho$.

6 Properties of Equivalence Relations

For a mapping $f\colon A \to B$, if we consider a relation $\{(x, y) \in A \times A \mid f(x) = f(y)\}$,

$$\begin{array}{ll} & f(x) = f(y) \\ \iff & \exists z \in B.((x, z) \in f \text{ and } (y, z) \in f) \\ \iff & \exists z \in B.((x, z) \in f \text{ and } (z, y) \in f^\sharp) \\ \iff & (x, y) \in ff^\sharp \ . \end{array}$$

So, we have $ff^\sharp = \{(x, y) \in A \times A \mid f(x) = f(y)\}$. This relation is a typical example of equivalence relations. It is not very difficult to show that $ff^\sharp$ is an equivalence relation. By the totality of f, reflexivity $\mathsf{id}_A \subseteq ff^\sharp$ is immediate. By the univalency of f, it holds that

$$(ff^\sharp)(ff^\sharp) = f(f^\sharp f)f^\sharp \subseteq f\mathsf{id}_B f^\sharp = ff^\sharp \ .$$

So, $ff^\sharp$ is transitive. Since $(ff^\sharp)^\sharp = f^{\sharp\sharp}f^\sharp = ff^\sharp$ holds, $ff^\sharp$ is symmetric.

From the above discussion, we learn a construction of an equivalence relation from a mapping. The next proposition shows the opposite direction.

Proposition 6.1 *For an equivalence relation $\rho\colon A \rightharpoonup A$ on A, there exists a mapping $f\colon A \to Y$ satisfying $ff^\sharp = \rho$.*

Proof. By Lemma 4.3, there exists a mapping $f\colon A \to \wp(A)$ such that $\rho = f \ni_A$ for an equivalence relation ρ. By the reflexivity of ρ and the univalency of f, we have

$$ff^\sharp \subseteq ff^\sharp\rho = ff^\sharp f \ni_A \subseteq f \ni_A = \rho \ .$$

By Proposition 3.4, there exist mappings u and v satisfying $\rho = u^\sharp v$. Since

$$\begin{array}{rcll} uf \ni_A & = & u\rho & (\text{by } f \ni_A = \rho) \\ & \subseteq & vv^\sharp u\rho & (\text{by totality of } v) \\ & = & v\rho^\sharp\rho & (\text{by } u^\sharp v = \rho) \\ & \subseteq & v\rho\rho & (\text{by symmetricity of } \rho) \\ & \subseteq & v\rho & (\text{by transitivity of } \rho) \\ & = & vf \ni_A & (\text{by } f \ni_A = \rho) \end{array}$$

$$\begin{array}{rcll} vf \ni_A & = & v\rho & (\text{by } f \ni_A = \rho) \\ & \subseteq & uu^\sharp v\rho & (\text{by totality of } u) \\ & = & v\rho\rho & (\text{by } u^\sharp v = \rho) \\ & \subseteq & v\rho & (\text{by transitivity of } \rho) \\ & = & vf \ni_A & (\text{by } f \ni_A = \rho) \end{array}$$

hold, we have $uf \ni_A = vf \ni_A$. By Lemma 4.2, $uf = vf$ holds. So,

$$\begin{array}{rcll} \rho & = & u^\sharp v & \\ & \subseteq & ff^\sharp u^\sharp vff^\sharp & (\text{by totality of } f) \\ & = & ff^\sharp u^\sharp uff^\sharp & (\text{by } vf = uf) \\ & \subseteq & ff^\sharp ff^\sharp & (\text{by univalency of } u) \\ & \subseteq & ff^\sharp & (\text{by univalency of } f) \end{array}$$

holds. Therefore, we have $\rho = ff^\sharp$.

The mapping $f\colon A \to \wp(A)$ given in Proposition 6.1 determines the equivalence class of $a \in A$ with respect to ρ.

Proposition 6.2 *For an equivalence relation $\rho\colon A \rightharpoondown A$ and a mapping $f\colon A \to \wp(A)$ satisfying $\rho = ff^\sharp$, the following holds.*

1. $a \in f(a)$,
2. $(a, b) \in \rho \iff f(a) = f(b)$,
3. $f(a) \neq f(b) \Rightarrow f(a) \cap f(b) = \emptyset$.

Proof. 1. By the reflexivity of ρ,

$$\begin{aligned}(a,a) \in \rho &\iff (a,a) \in f \ni_A \\ &\iff \exists S \subseteq A.((a,S) \in f \text{ and } (S,a) \in \ni_A) \\ &\iff \exists S \subseteq A.((a,S) \in f \text{ and } a \in S) \\ &\iff a \in f(a)\end{aligned}$$

2. We have already shown at the beginning of this chapter.
3. By 2, $f(a) \neq f(b) \iff (a,b) \notin \rho$ holds. Assume that $f(a) \cap f(b) \neq \emptyset$. Then

$$\begin{aligned}& \exists c \in A.((a,c) \in f \ni_A \text{ and } (b,c) \in f \ni_A) \\ \iff & (a,b) \in (f \ni_A)(f \ni_A)^\sharp \ .\end{aligned}$$

Since $(f \ni_A)(f \ni_A)^\sharp = \rho\rho^\sharp \subseteq \rho\rho \subseteq \rho$, $(a,b) \in \rho$. This is contradiction to $f(a) \neq f(b)$.

For an equivalence relation $\rho\colon A \rightharpoondown A$, it is known that there exists natural surjection called canonical surjection from A to the quotient set of A by ρ. Since the canonical surjection often plays an important rôle in universal algebras, we cite this briefly.

Proposition 6.3 *For an equivalence relation $\rho\colon A \rightharpoondown A$ on A, there exists a surjection $p\colon A \to Q$ such that $pp^\sharp = \rho$.*

Proof. By Proposition 6.1, there exists a mapping $f\colon A \to \wp(A)$ satisfying $ff^\sharp = \rho$. Also by theorem 3.8, there exist a surjection $p\colon A \to Q$ and an injection $m\colon Q \to B$ for $f\colon A \to \wp(A)$. Since m is injective, $ff^\sharp = (pm)(pm)^\sharp = pmm^\sharp p^\sharp = pp^\sharp$.

Q and $p\colon A \to Q$ in Proposition 6.3 correspond to the quotient set of A by ρ and the canonical surjection, respectively.

7 Conclusion

We have studied basic definitions and properties of binary relations. The author believe that we have studied on equivalence relations more carefully than usual. Since more technical preparation is needed, we have left it at recalling the definition of orderings.

Could you feel closer to binary relations? For people who wants to study relations more, [3] is recommended. This book provides quite wide knowledge from basics to applications in computer science.

We have seen point-free formalisations and proofs at some parts of this note. [1] and [2] introduce a kind of categories called allegories. In these books we can learn benefit and beauty of point-free relational calculus.

This note is based on what the author studied in his master course under the instruction of Prof. Kawahara at Kyushu University.

References

[1] P.J. Freyd and A. Scedrov, Categories, Allegories (North-Holland, 1990).

[2] P.T. Johnstone, Sketches of an Elephant: A Topos Theory Compendium, Vol.1 (Oxford University Press, 2002).

[3] G. Schmidt and T. Ströhlein, Relations and Graphs (Springer-Verlag, 1993).