円環グラフ上の追跡回避ゲームの解析とシミュレーション

池田 有希

九州大学大学院数理学府

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本研究は九州大学の深井康成先生，溝口佳寛先生との共同研究である

1. Introduction

Hunter vs. Rabbit game の解析を行った.

- このゲームはハンター，ラビットと呼ばれる二人のプレイヤーにより行われる.
- プレイヤーは決められた領域内を移動する.
- ハンターがラビットを捕まえた時，ゲームは終了する.
- 二人のプレイヤーはお互いに現在の位置は分からない.
- またお互いの移動方針もわからない.

このゲームを円環グラフ上のランダムウォークを使い，調査を行う.

グラフについて

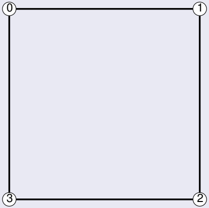
グラフ G は頂点集合 V と辺集合 E によって

$$G = (V, E)$$

と定義される．

例

$V = \{0, 1, 2, 3\}$， $E = \{(0, 1), (1, 2), (2, 3), (3, 0)\}$ であるとき，グラフ $G = (V, E)$ は以下のように図示できる．

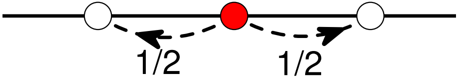


グラフ上のランダムウォークについて

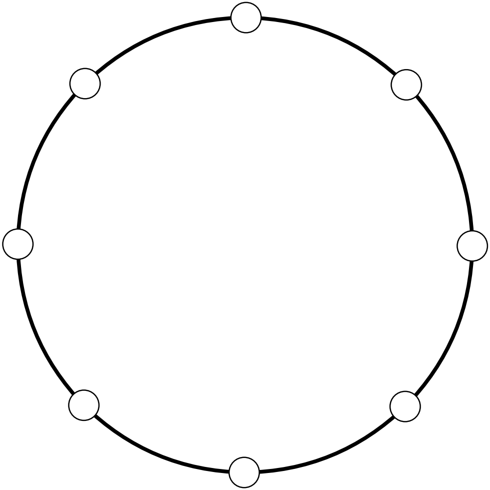
X_t を時間 t で移動するグラフ上の距離とする．
このとき

$$P\{X_t = k\} = \begin{cases} \frac{1}{2} & (k = \pm 1) \\ 0 & (k \neq \pm 1) \end{cases}$$

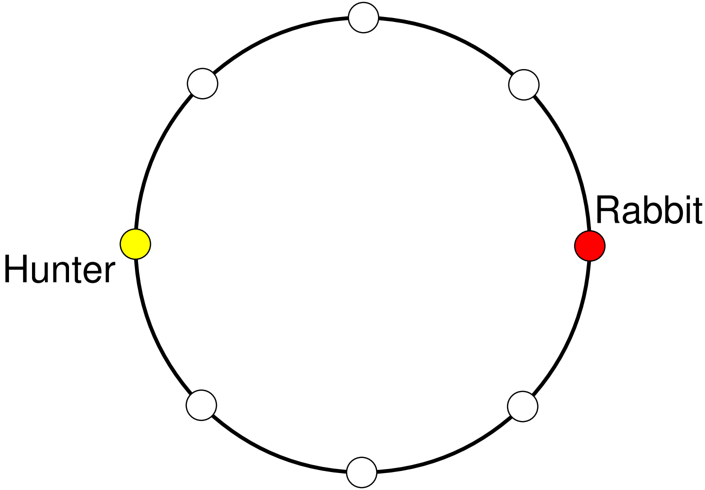
とする．



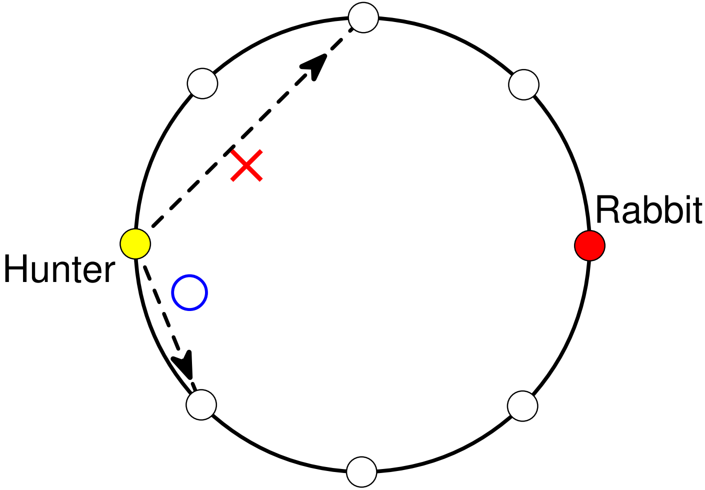
ゲームは円環グラフ上で行う



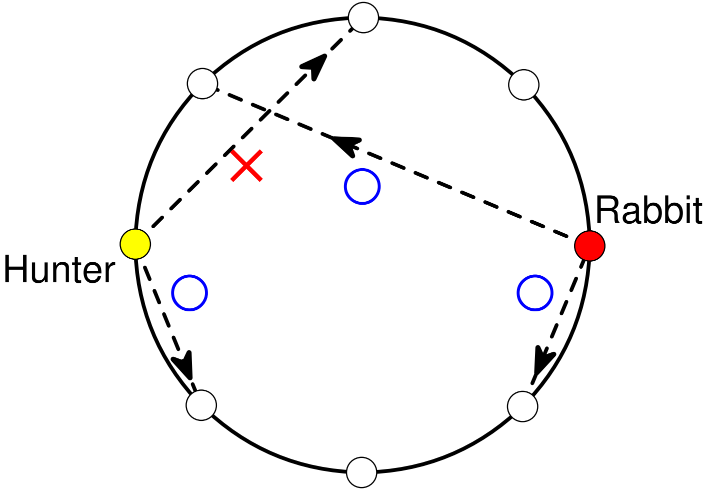
初期位置を決定する



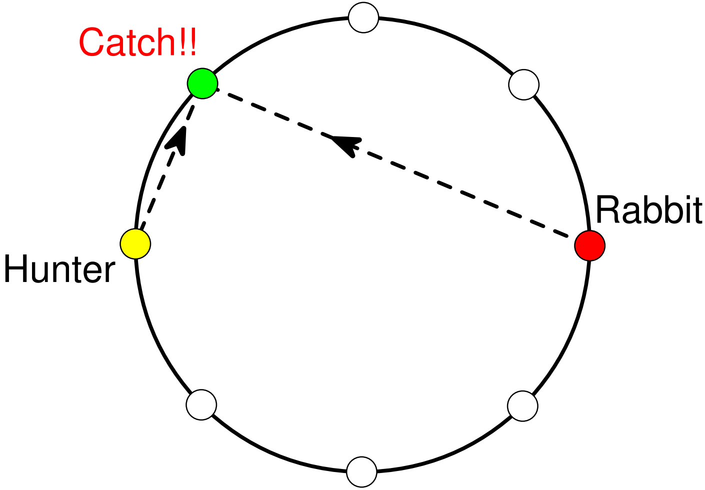
ハンターの移動に関して



ラビットの移動に関して



ハンターがラビットを捕まえる



応用

- Hunter vs Rabbit game はモバイルアドホックネットワークの通信モデルの解析に使われる [I. Chatzigiannakis et al. 2001].
- モバイルアドホックネットワークは携帯電話間のメール送受信に関係している.
  - ハンター ··· 基地局間の通信
  - ラビット ··· 携帯電話のユーザー
- 携帯電話から携帯電話へメールを送信し，受信するまでの時間の期待値は，ハンターがラビットを捕まえるまでの時間の期待値に一致することがわかっている.

Domain Difference

Princess-Monster game

The difference of the Hunter vs Rabbit game

The Monster catches the Princess if distance of two players smaller than the some value.

R. Isascs(1965)

R. Isaccs analyzed this game on a cycle graph.

S. Gal(1976)

S. Gal analyzed this game on convex multidimensional domain.

↓

After that, this game has been investigated by Alpern, Zelekin, and so on.

Players Difference

Deterministic pursuit-evasion game

The difference of the Hunter vs Rabbit game

- We consider a runaway hide a dark spot, for example a tunnel.
- The number of players is greater than the Hunter vs Rabbit game.

T. D. Parsons(1976, 1978)

Parsons innovated the search number of a graph.

N. Megiddo et al. (1988)

Computation time of the search number of a graph is NP-hard.

Investigation of Hitting time(1)

Hunter vs Rabbit game

This game is studied about the expected value of rounds that the hunter catches the rabbit.

R. Aleiunas et al.(1979)

- $O(nm^2)$

M. Adler et al.(2003)

- $O(n\log(diam(G))$ for any graph, if the hunter chooses a good strategy.
- $\Omega(n\log(diam(G))$ for a special graph, if the rabbit chooses a good strategy.

Investigation of Hitting time(2)

Y. Ikeda(2013)

- $16n \log(diam(G)) + 24n$ for any graph, if the hunter chooses a good strategy.
- $n(688 + \log(diam(G))/2)/1376$ for a special graph, if the rabbit chooses a good strategy.

Our formulation

- We propose three assumptions for a strategy of the rabbit.
- We have the general lower bound formula of a probability that the hunter catches the rabbit.
- The strategy of the rabbit is formalized using a one dimensional random walk over $\mathbb{Z}$.

Our result(1)

- ラビットの戦略に依存するパラメータ β の値により，ハンターがラビットを捕まえる確率の下限の分類を行った．
- $\beta \in (0,1)$ であるとき，下限は定数となる．
- $\beta = 1$ であるときは $(\frac{1}{c_* \pi} \log N + c_2)^{-1}$ となる．ただし c_2 と c_* は定数である．
- $\beta \in (1,2]$ であるときは $c_4 N^{(1-\beta)/\beta}$ となる．ただし c_4 は定数である．

Our result(2)

- We develop a simulation program to simulate the Hunter vs Rabbit game using C++.
- We confirm our bounds formula, and asymptotic behavior of those bounds by results of simulations.

Our result(3)

- We show the transition probability for given strategies.
- In our program, we compute random walks using the digamma function.
- To simulate the Hunter vs Rabbit game, we have to use the digamma function.
- We also summarize the property of the digamma function and the relation between transition matrices of random walks and the digamma function.
- We show experimental results with our program.

2. Definition of Random Walks

ランダムウォークの定義について

プレイヤーごとに以下を定義する.

- 初期位置の決定
- 一次元ランダムウォーク

→ プレイヤーの戦略を定義する.

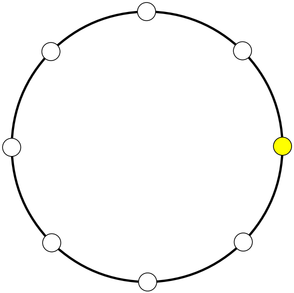
さらにハンターがラビットを捕まえる事を定義する.

以下，円環グラフの頂点集合を $V_N = \{0, 1, 2 \ldots, N-1\}$ とする.

Definition of random walks of the rabbit(1)

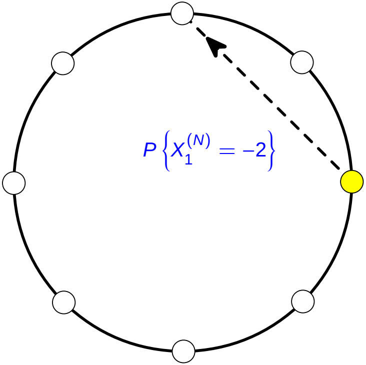
- Let $N \in \mathbb{N}$ be fixed and $V_N = \{0, 1, 2, \ldots, N-1\}$.
- We denote by $X_0^{(N)}$ a random variable defined on a probability space $(\Omega_N, \mathcal{F}_N, \mu_N)$ taking values in V_N with

$$\mu_N \left\{ X_0^{(N)} = l \right\} = \frac{1}{N} \quad (l \in V_N).$$



Definition of random walks of the rabbit(2)

- Let $X_1^{(N)}, X_2^{(N)}, \ldots$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in the integer lattice $\mathbb{Z}$.



Definition of random walks of the rabbit(3)

- Let $X_1^{(N)}, X_2^{(N)}, \ldots$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in the integer lattice $\mathbb{Z}$.
- A one-dimensional random walk $\{S_n\}_{n=1}^{\infty}$ is defined by

$$S_n = \sum_{j=1}^{n} X_j^{(N)}.$$

- A rabbit's strategy $\left\{\mathcal{R}_n^{(N)}\right\}_{n=0}^{\infty}$ is defined by

$$\mathcal{R}_0^{(N)} = X_0^{(N)} \quad \text{and} \quad \mathcal{R}_n^{(N)} = \left(X_0^{(N)} + S_n \mod N\right).$$

- $\mathcal{R}_n^{(N)}$: the position of the rabbit at time n on V_N.

Definition of random walks of the hunter

- Let $Y_1, Y_2, \dots$ be independent, identically distributed random variables defined on a probability space $(\Omega_{\mathcal{H}}, \mathcal{F}_{\mathcal{H}}, P_{\mathcal{H}})$ taking values in the integer lattice $\mathbb{Z}$ with

$$P_{\mathcal{H}}\{|Y_1| \le 1\} = 1.$$

- Hunter's strategy $\left\{\mathcal{H}_n^{(N)}\right\}_{n=0}^{\infty}$ is defined by

$$\mathcal{H}_0^{(N)} = 0 \text{ and } \mathcal{H}_n^{(N)} = \left(\sum_{j=1}^{n} Y_j \mod N\right).$$

- $\mathcal{H}_n^{(N)}$: the position of the hunter at time n on V_N.

The hunter catches the rabbit

- Put

$$\mathbb{P}_{\mathcal{R}}^{(N)} = \mu_N \times P \quad \text{and} \quad \tilde{\mathbb{P}}^{(N)} = P_{\mathcal{H}} \times \mathbb{P}_{\mathcal{R}}^{(N)}.$$

- We will discuss the probability that the hunter catches the rabbit until time N on V_N, that is,

$$\tilde{\mathbb{P}}^{(N)}\left(\bigcup_{n=1}^{N} \left\{ \mathcal{H}_n^{(N)} = \mathcal{R}_n^{(N)} \right\} \right).$$

- We investigate the asymptotic estimate of this probability as $N \to \infty$.

3. Lower bounds of probabilities
that the hunter catches the rabbit

We define conditions (A1), (A2) and (A3) as follows.

(A1) A random walk is strongly aperiodic, i.e. for each $y \in \mathbb{Z}$, the smallest subgroup containing the set

$$\left\{ y + k \in \mathbb{Z} \mid P\left\{ X_1^{(N)} = k \right\} > 0 \right\}$$

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(A2) $P\left\{ X_1^{(N)} = k \right\} = P\left\{ X_1^{(N)} = -k \right\} \quad (k \in \mathbb{Z})$,

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(A2) $P\left\{X_1^{(N)} = k\right\} = P\left\{X_1^{(N)} = -k\right\} \quad (k \in \mathbb{Z})$,

(A3) There exist $\beta \in (0, 2]$, $c_* > 0$ and $\varepsilon > 0$ such that

$$\phi(\theta) := \sum_{k \in \mathbb{Z}} e^{i\theta k} P\left\{X_1^{(N)} = k\right\} = 1 - c_* |\theta|^\beta + O\left(|\theta|^{\beta+\varepsilon}\right).$$

Remark.

Let

$$P_\alpha\{X_t = k\} = \begin{cases} \dfrac{1}{a|k|^\alpha} & (k \neq 0) \\ 1 - \dfrac{2}{a}\displaystyle\sum_{l=1}^{\infty}\dfrac{1}{|l|^\alpha} & (k = 0). \end{cases}$$

Then, $\beta = \alpha - 1$ for $\alpha \in (1, 3]$. When $\beta_1 \leq \beta_2$ where $\beta_1, \beta_2 \in (0, 2]$,

$$P_{\beta_1+1}\{X_t = k\} \geq P_{\beta_2+1}\{X_t = k\}.$$

Main theorem

Theorem 1

Assume that X_1 *satisfies* $(A1)-(A3)$. *If* $\beta \in (0,2]$, *then there exists constants* $c_1 > 0$, $c_2 > 0$ *and* $c_4 > 0$ *such that for* $N \in \mathbb{N} \setminus \{1\}$,

$$\mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)} = (y_n \mod N)\right\}\right) \geq \begin{cases} c_1 & (\beta \in (0,1)), \\ \dfrac{1}{\frac{1}{c_*\pi}\log N + c_2} & (\beta = 1), \\ \dfrac{c_4}{N^{(\beta-1)/\beta}} & (\beta \in (1,2]). \end{cases}$$

Proposition 1

For $N \in \mathbb{N} \setminus \{1\}$,

$$\frac{1}{\sum_{i=0}^{N-1} p_i^{(N)}} \le \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)} = (y_n \mod N)\right\}\right)$$

where $[y]_N = \{y + kN \mid k \in \mathbb{Z}\}$ *and*

$$p_i^{(N)} = \begin{cases} 1 & (i = 0), \\ \max_{|y| \le i,\ y \in \mathbb{Z}} P\{S_i \in [y]_N\} & (i \in \mathbb{N}). \end{cases}$$

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Proposition 2

If a one-dimensional random walk satisfies $(A2)$,

$$P\{S_n \in [l]_N\} = \frac{1}{N} + \frac{2}{N} \sum_{1 \le j \le (N-1)/2} \phi^n\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi}{N}l\right) + J_N(n,l),$$

where

$$J_N(n,l) = \begin{cases} \frac{1}{N}\phi^n(\pi)\cos(\pi l) & (\text{ if } N \text{ is even }), \\ 0 & (\text{ if } N \text{ is odd }). \end{cases}$$

Proposition 3

Assume (A1) – (A3). Then there exists a constant $c_8 > 0$, $c_9 > 0$ *and* $c_{10} > 0$ *such that*

$$\sum_{i=0}^{N-1} p_i^{(N)} \le \begin{cases} c_8 & (\beta \in (0,1)), \\ \dfrac{1}{c_* \pi} \log N + c_9 & (\beta = 1), \\ c_{10} N^{(\beta-1)/\beta} & (\beta \in (1,2]). \end{cases}$$

From Proposition 1,

$$\frac{1}{\sum_{i=0}^{N-1} p_i^{(N)}} \le \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = (y_n \mod N) \right\} \right).$$

Proof of Proposition 3 (1)

$r \in (0,\pi)$ とする．$(A3)$ より，

$$\phi(\theta) = 1 - c_*|\theta|^\beta + O(|\theta|^{\beta+\varepsilon}).$$

$|\theta| < r$ に対し，

$$\left|\phi(\theta) - (1 - c_*|\theta|^\beta)\right| \le C_*|\theta|^{\beta+\varepsilon}$$

となるように定数 C_* をとることができる．さらに

$$C_* r_*^\varepsilon \le \frac{1}{2}c_* \text{ and } c_* r_*^\beta \le \frac{1}{3}$$

となるように十分小さく $r_* \in (0, r]$ をとることができる．このとき $|\theta| < r_*$ に対して，

$$0 \le \frac{1}{2}c_*|\theta|^\beta \le \left|1 - \phi(\theta)\right| \le \frac{3}{2}c_*|\theta|^\beta \le \frac{1}{2}.$$

Proof of Proposition 3 (2)

In 1967, F. Spitzer showed that a strongly aperiodic random walk (A1) has the property that $\left|\phi(\theta)\right| = 1$ only when θ is a multiple 2π. From the definition of $\phi(\theta)$, $\left|\phi(\theta)\right|$ is a continuous function on the bounded close set $[-\pi, -r_*] \cup [r_*, \pi]$, and $\left|\phi(\theta)\right| \leq 1$ for $\theta \in [-\pi, \pi]$. Hence, there exists a $\rho_* < 1$, depending on $r_* \in (0, \pi]$, such that

$$\max_{r_* \leq |\theta| \leq \pi} \left|\phi(\theta)\right| \leq \rho_*.$$

From $(A2)$, $\phi(\theta)$ takes a real number. Therefore, from $0 \leq |1 - \phi(\theta)| \leq 1/2$, for $\theta \in (-r_*, 0) \cup (0, r_*)$,

$$\frac{1}{2} < \phi(\theta) = \left|\phi(\theta)\right| < 1.$$

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$$\frac{1}{2} < \phi(\theta) = \left|\phi(\theta)\right| < 1.$$

Proof of Proposition 3 (3)

Let N is even.

Then, from Proposition 2,

$$P\{S_i \in [l]_N\} = \frac{1}{N} + \frac{2}{N}\sum_{1\le j\le (N-1)/2} \phi^i\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi}{N}l\right) + \frac{1}{N}\phi^i(\pi)\cos(\pi l).$$

From $\cos(\theta) \le 1$ for any $\theta \in \mathbb{R}$,

$$\begin{aligned} p_i^{(N)} &= \max_{|l|\le i} P\{S_i \in [l]_N\} \\ &\le \frac{1}{N} + \frac{2}{N}\sum_{1\le j\le (N-1)/2} \phi^i\left(\frac{2j\pi}{N}\right) + \frac{1}{N}\phi^i(\pi). \end{aligned}$$

Proof of Proposition 3 (4)

From $\max_{r_* \leq |\theta'| \leq \pi} |\phi(\theta')| \leq \rho_*$,

$$\phi^i(\pi) \leq \rho_*^i$$

and

$$\sum_{(r_*/(2\pi))N \leq j \leq (N-1)/2} \phi^i\left(\frac{2j\pi}{N}\right) \leq \frac{N-1}{2}\rho_*^i.$$

Then,

$$\begin{aligned} p_i^{(N)} &\leq \frac{1}{N} + \frac{2}{N} \sum_{1 \leq j \leq (N-1)/2} \phi^i\left(\frac{2j\pi}{N}\right) + \frac{1}{N}\phi^i(\pi) \\ &\leq \frac{1}{N} + \frac{2}{N} \sum_{1 \leq j < (r_*/(2\pi))N} \phi^i\left(\frac{2j\pi}{N}\right) + \rho_*^i. \end{aligned}$$

Proof of Proposition 3 (5)

Then

$$\sum_{i=0}^{N-1} p_i^{(N)} \leq 1 + \Phi_N + \frac{1}{1-\rho_*},$$

where

$$\Phi_N = \sum_{1 \leq j < (r_*/(2\pi))N} \frac{2}{N} \frac{1 - \left|\phi\left(\frac{2j\pi}{N}\right)\right|^N}{1 - \left|\phi\left(\frac{2j\pi}{N}\right)\right|}.$$

From $|\phi(\theta)| < 1$ for any $|\theta| < r_*$,

$$\Phi_N \leq \sum_{1 \leq j < (r_*/(2\pi))N} \frac{2}{N} \frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)}.$$

Proof of Proposition 3 (6)

We consider the case of $\beta \in (0,1]$. We decompose the right-hand side of the above to obtain

$$\sum_{1 \leq j < (r_*/(2\pi))N} \frac{2}{N} \frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)} = \tilde{\Phi}_N + E_N,$$

where

$$\tilde{\Phi}_N = \frac{2^{1-\beta}}{\pi^\beta c_*} N^{\beta-1} \sum_{1 \leq j < (r_*/(2\pi))N} j^{-\beta},$$

$$E_N = \sum_{1 \leq j < (r_*/(2\pi))N} \frac{2}{N} \left(\frac{1}{1 - \phi\left(\frac{2j\pi}{N}\right)} - \frac{1}{c_* \left(\frac{2j\pi}{N}\right)^\beta} \right).$$

Proof of Proposition 3 (7)

Let $c_{11} = \frac{2^{2+\varepsilon-\beta}\pi^{\varepsilon-\beta}C_*}{c_*^2}$. Then,

$$|E_N| \le c_{11}/(1+\varepsilon-\beta).$$

It is easy to see that

$$\tilde{\Phi}_N \le \begin{cases} \dfrac{2^{1-\beta}}{\pi^\beta c_*(1-\beta)} & (\beta \in (0,1)) \\ \dfrac{1}{\pi c_*}\log N + \dfrac{1}{\pi c_*} & (\beta = 1). \end{cases}$$

Proof of Proposition 3 (8)

When $\beta \in (1, 2]$,

$$\Phi_N \le \Phi_N^{(1)} + \Phi_N^{(2)},$$

where $N(\beta) = \min \left\{ N^{(\beta-1)/\beta}, (r_*/(2\pi))N \right\}$ and

$$\Phi_N^{(1)} = \sum_{1 \le j < N(\beta)} \frac{2}{N} \frac{\left| 1 - \phi^N\left(\frac{2j\pi}{N}\right) \right|}{\left| 1 - \phi\left(\frac{2j\pi}{N}\right) \right|},$$

$$\Phi_N^{(2)} = \sum_{N(\beta) \le j < (r_*/(2\pi))N} \frac{2}{N} \frac{1}{\left| 1 - \phi\left(\frac{2j\pi}{N}\right) \right|}.$$

Proof of Proposition 3 (8)

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where $N(\beta) = \min\left\{N^{(\beta-1)/\beta}, (r_*/(2\pi))N\right\}$ and

$$\Phi_N^{(1)} = \sum_{1 \leq j < N(\beta)} \frac{2}{N} \frac{\left|1 - \phi^N\left(\frac{2j\pi}{N}\right)\right|}{\left|1 - \phi\left(\frac{2j\pi}{N}\right)\right|} \leq N^{(\beta-1)/\beta},$$

$$\Phi_N^{(2)} = \sum_{N(\beta) \leq j < (r_*/(2\pi))N} \frac{2}{N} \frac{1}{\left|1 - \phi\left(\frac{2j\pi}{N}\right)\right|}.$$

Proof of Proposition 3 (9)

For $|\theta| < r_*$,

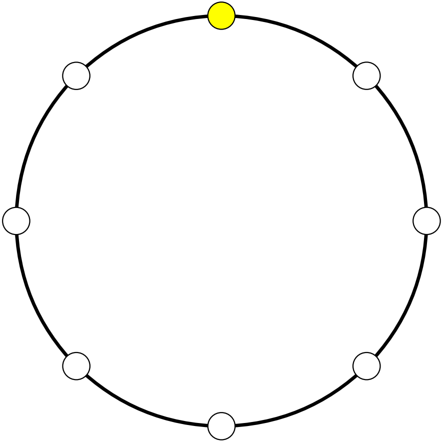
$$\frac{1}{2}c_*|\theta| \le |1 - \phi(\theta)|.$$

Therefore,

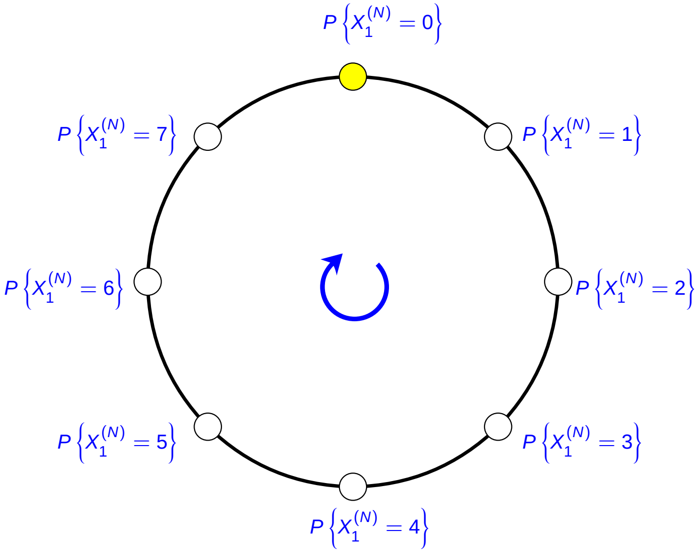
$$\begin{aligned}
\Phi_N^{(2)} &= \sum_{N(\beta)\le j<(r_*/(2\pi))N} \frac{2}{N} \frac{1}{\left|1 - \phi\left(\frac{2j\pi}{N}\right)\right|} \\
&\le \frac{2^{2-\beta}}{c_*\pi^\beta} N^{\beta-1} \left(\sum_{N(\beta)\le j<(r_*/(2\pi))N} j^{-\beta} \right) \\
&\le \frac{2^{2-\beta}}{c_*\pi^\beta} \left(1 + \frac{1}{\beta - 1}\right) N^{(\beta-1)/\beta}.
\end{aligned}$$

4. Transition Matrix
of the Random Walks on a Cycle Graph

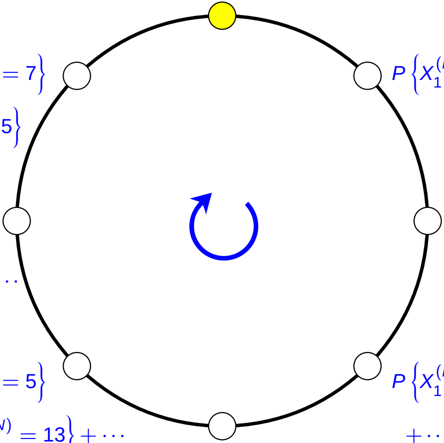
Transition Probability(1)



Transition Probability(1)

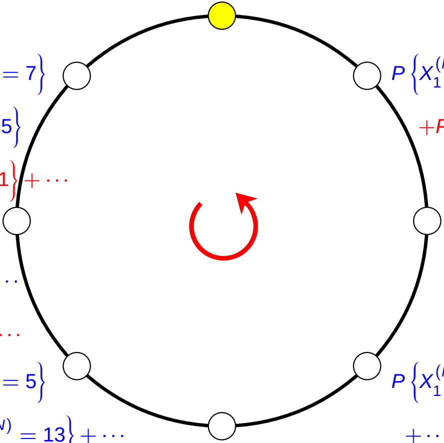


Transition Probability(1)



Transition Probability(1)

$$P\left\{X_1^{(N)}=0\right\}+P\left\{X_1^{(N)}=8\right\}+\cdots+P\left\{X_1^{(N)}=-8\right\}+\cdots$$



$$P\left\{X_1^{(N)}=7\right\}+P\left\{X_1^{(N)}=15\right\}+\cdots+P\left\{X_1^{(N)}=-1\right\}+\cdots$$

$$P\left\{X_1^{(N)}=1\right\}+P\left\{X_1^{(N)}=9\right\}+\cdots+P\left\{X_1^{(N)}=-7\right\}+\cdots$$

$$P\left\{X_1^{(N)}=6\right\}+P\left\{X_1^{(N)}=14\right\}+\cdots+P\left\{X_1^{(N)}=-2\right\}+\cdots$$

$$P\left\{X_1^{(N)}=2\right\}+P\left\{X_1^{(N)}=10\right\}+\cdots+P\left\{X_1^{(N)}=-6\right\}+\cdots$$

$$P\left\{X_1^{(N)}=5\right\}+P\left\{X_1^{(N)}=13\right\}+\cdots+P\left\{X_1^{(N)}=-3\right\}+\cdots$$

$$P\left\{X_1^{(N)}=3\right\}+P\left\{X_1^{(N)}=11\right\}+\cdots+P\left\{X_1^{(N)}=-5\right\}+\cdots$$

$$P\left\{X_1^{(N)}=4\right\}+P\left\{X_1^{(N)}=12\right\}+\cdots+P\left\{X_1^{(N)}=-4\right\}+\cdots$$

Transition Probability(2)

A transition matrix $\mathbb{P}(N, P)$ is defined by

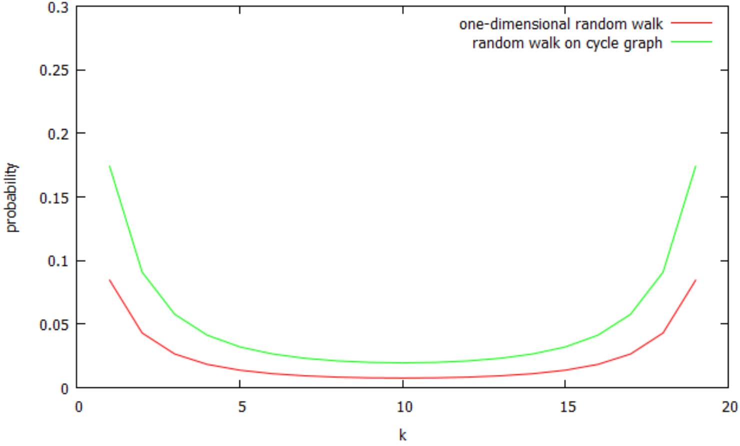
$$\mathbb{P}(N, P) = \left\{\mathbb{P}_{i,j}\right\}_{0 \leq i,j \leq N-1}$$

where

$$\begin{aligned}\mathbb{P}_{i,i+d} &= \mathbb{P}_{(i \mod N),(i+d \mod N)} \\ &= \sum_{k=0}^{\infty}(P\{x_t = kN + d\} + P\{x_t = d - (k+1)N\}).\end{aligned}$$

For example, when we simulate the Hunter vs Rabbit game using a computer,

$$\mathbb{P}_{i,i+d} = \sum_{k=0}^{M}(P\{x_t = kN + d\} + P\{x_t = d - (k+1)N\}).$$



$$P\{X_t = k\} = \begin{cases} \frac{1}{2(k+1)(k+2)} & (k \neq 0) \\ \frac{1}{2} & (k = 0) \end{cases}$$

Digamma function

The digamma function $\psi(x)$ is defined by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x)$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

is the gamma function.

5. Computer Simulation

- In this section, we show some experimental results about the Hunter vs Rabbit game on a cycle graph.
- We compute $P\{(S_n \mod N) = k\}$ by using the digamma function and the class `discrete_distribution` in C++.
- We can show the probability of the hunter catches the rabbit and the expected value of the time until the hunter catches the rabbit using this application.

Input

- `vertex`: the number of vertices
- `strategy_h`: a strategy of the hunter
- `strategy_r`: a strategy of the rabbit
- `loop`: the number of trials

Output

- average of a time that the hunter catches the rabbit
- average of a probability that the hunter catches the rabbit

main 関数

```
int main()
{
    /*
    乱数
    */
    srand((unsigned)time(NULL));

    /*
    シミュレーション確率と下限のプロット
    */
    int vertex = 500.0;
    double sp = 1.0;
    int h_0 = 0;
    double speed = 0.0;
    int strategy_h = RANDOM_SPEED_STRATEGY;
    int strategy_r = HEAVY_TAILED_RANDOM_WALK_GAMMA;
    int loop = 1000;
    probability_plot(vertex, sp, h_0, speed, strategy_h, strategy_r, loop);

    return 0;
}
```

呼び出される関数

```
void probability_plot(int vertex, double sp, int h_0, double speed, int strategy_h, int strategy_r, int loop)
{
    std::cout << "Simulation start" << std::endl;
    ofstream fout("prop.dat");
    int h = h_0;
    double p_avg = 0.0;

    //simulation sim(vertex, h, r_0, speed, sp, strategy_h, strategy_r);

    for (int r_0 = 1; r_0 < vertex; r_0++)
    {
    //std:cout << "loop " << r_0 << std::endl;
        simulation sim(vertex, h, r_0, speed, sp, strategy_h, strategy_r);
        double p = sim.probability(loop);
        p_avg += p;
        fout << r_0 << " " << p << endl;
    }

    p_avg = p_avg / vertex;
    std::cout << "average" << p_avg << std::endl;

    double lower = lower_bound(vertex, strategy_r, sp);


    std::cout << "lower^-1:" << 1.0 / lower << std::endl;

    std::cout << "average/lower:" << p_avg / lower << std::endl;

    gnuplot::CGnuplot gp;

    gp.SetTitle("The probability of that the Rabbit is caught");
    gp.Command("set key top");
    gp.SetLabel("a first position of the Rabbit", "the probability that the rabbit is caught");
    gp.SetXRange(0, vertex);
    gp.SetYRange(0, 1);
    gp.Command("lower = %f", lower);
    gp.Command("avg=%f", p_avg);
    gp.Command("plot 'prop.dat' with lines title 'a probability that the Rabbit is caught', lower with lines t
    gp.DumpToEps("result");
    __KEYWAIT__;
}
```

rabbit クラスのサブクラス

```
class HeavyTailedRandomWalkG
{
public:
    double operator()(double k);
    double sp;//パラメータa
    double v;//円環グラフの頂点数
    HeavyTailedRandomWalkG();//コンストラクタ
    HeavyTailedRandomWalkG(double d, int n);//コンストラクタ, d:移動距離, n:頂点数
    double cot(double rad);//cot関数
};
```

遷移行列の定義

```
double Rabbit::HeavyTailedRandomWalkG::operator()(double k)
{
    double ss = 0.0;

    if (k == 0){
        for (double m = 1.0; m < v; m++){
            ss += (cos(4.0 * M_PI* m / v) - cos(2.0 * M_PI * m / v))*log(2.0 - 2.0 * cos(2.0 * M_PI * m / v));
        }
        return (M_PI * (cot(M_PI / v) - cot(2.0 * M_PI / v)) / 2.0 + ss / 2.0 - v / 2.0) / (sp*v) + 1.0 - (v*v + 3.0*v + 3.0) / (2.0*sp*(v + 1

    }
    else if (k == 1){
        for (double m = 1.0; m < v; m++){
            ss += (cos(6.0 * M_PI* m / v) - cos(4.0 * M_PI * m / v) + cos(2.0 * M_PI * m / v))*log(2.0 - 2.0 * cos(2.0 * M_PI * m / v));
        }
        return (M_PI * (-cot(3.0 * M_PI / v) + cot(2.0 * M_PI / v) - cot(M_PI / v)) / 2.0 - log(v) + v + ss / 2.0) / (sp*v);

    }
    else if (k == 2){
        for (double m = 1.0; m < v; m++){
            ss += (cos(8.0 * M_PI* m / v) - cos(6.0 * M_PI * m / v) - cos(2.0 * M_PI * m * (v - 1.0) / v))*log(2.0 - 2.0 * cos(2.0 * M_PI * m
        }
        return (M_PI * (-cot(4.0 * M_PI / v) + cot(3.0 * M_PI / v) + cot((v - 1.0) * M_PI / v)) / 2.0 + log(v) + ss / 2.0) / (sp*v);

    }
    else if (k == v - 2){
        for (double m = 1.0; m < v; m++){
            ss += (-cos(2.0 * M_PI* m * (v - 1.0) / v) + cos(8.0 * M_PI * m / v) - cos(6.0 * M_PI * m * (v - 1) / v))*log(2.0 - 2.0 * cos(2.0
        }
        return (M_PI * (cot(M_PI*(v - 1.0) / v) - cot(4.0 * M_PI / v) + cot(3.0 * M_PI / v)) / 2.0 + log(v) + ss / 2.0) / (sp*v);

    }
    else if (k == v - 1){
        for (double m = 1.0; m < v; m++){
            ss += (cos(2.0 * M_PI* m / v) + cos(6.0 * M_PI * m / v) - cos(4.0 * M_PI * m / v))*log(2.0 - 2.0 * cos(2.0 * M_PI * m / v));
        }
        return (M_PI * (-cot(M_PI / v) - cot(3.0 * M_PI / v) + cot(2.0 * M_PI / v)) / 2.0 - log(v) + v + ss / 2.0) / (sp*v);
    }
    else{
        for (double m = 1.0; m < v; m++){
            ss += (cos(2.0 * M_PI* m * (k + 2.0) / v) - cos(2.0 * M_PI * m * (k + 1.0) / v) + cos(2.0 * M_PI * m * (k - 2.0) / v) - cos(2.0 *
        }
```

Example 1. Heavy-tailed random walk

(Adler et al. 2003)

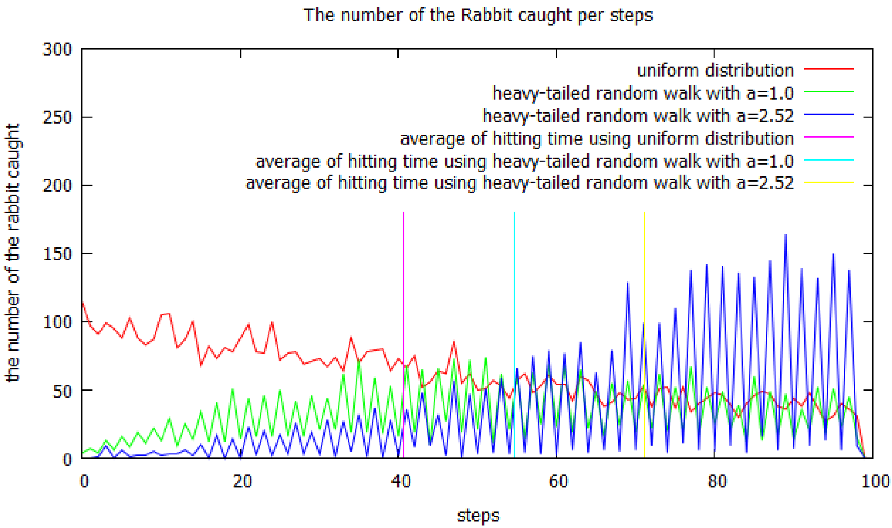
Let

$$P\{x_t = k\} = \begin{cases} \dfrac{1}{2a(|k|+1)(|k|+2)} & k \neq 0, \\ 1 - \dfrac{1}{2a} & k = 0, \end{cases}$$

where $a \geq \frac{1}{2}$.

In this case, $\beta = 1$, $c_* = \frac{\pi}{2a}$ and $\varepsilon = \frac{1}{2}$.

Result: Optimal parameter



Lower bound

$a = 1$ の場合，Proposition 3 の証明より $C_* = 1.225$, $\rho_* = 0.785802$ となる．したがって

$$\sum_{i=0}^{N-1} p_i^{(N)} \leq \frac{2}{\pi^2} \log N + 6.50503.$$

Proposition 1 より

$$\left(\frac{2}{\pi^2} \log N + 6.50503 \right)^{-1} \leq \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = (y_n \mod N) \right\} \right).$$

Transition matrix

We calculate elements of a transition matrix. For $d \neq 0$,

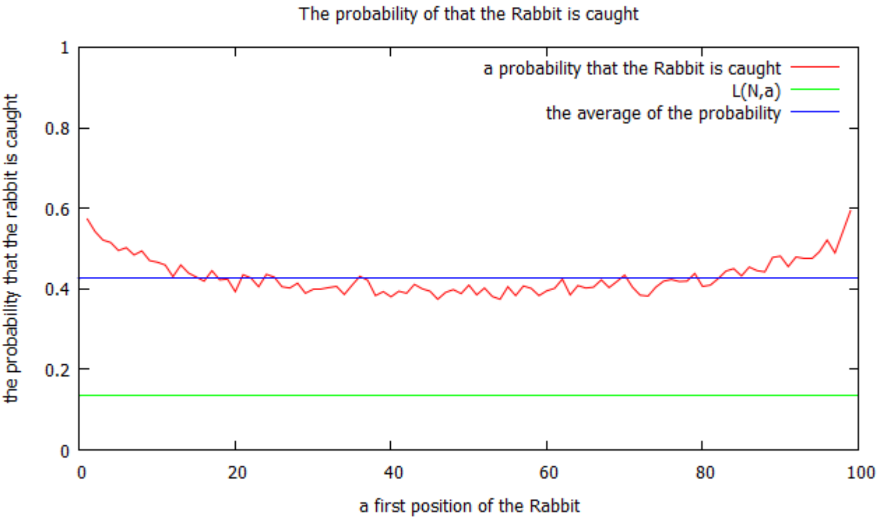
$$
\begin{aligned}
\mathbb{P}_{i,i} &= \frac{1}{aN}\left(\psi\left(\frac{N+2}{N}\right) - \psi\left(\frac{N+1}{N}\right)\right) \\
&\quad - \frac{1}{a(N+1)(N+2)} + 1 - \frac{1}{2a}, \\
\mathbb{P}_{i,i+d} &= \frac{1}{aN}\left(\psi\left(\frac{d+2}{N}\right) - \psi\left(\frac{d+1}{N}\right)\right. \\
&\quad \left. + \psi\left(\frac{N-d+2}{N}\right) - \psi\left(\frac{N-d+1}{N}\right)\right).
\end{aligned}
$$

Transition matrix

Theorem 4.1. $n > 4$, $2 < d < n-2$ とする．このとき

$$
\begin{aligned}
P_{i,i+n-1} &= \frac{1}{an}\left(\frac{\pi}{2}\left(-\cot\frac{\pi}{n} - \cot\frac{3\pi}{n} + \cot\frac{2\pi}{n}\right) - \log n + n\right.\\
&\quad \left. + \frac{1}{2}\sum_{m=1}^{n-1}\left(\cos\frac{2\pi m}{n} + \cos\frac{6\pi m}{n} - \cos\frac{4\pi m}{n}\right)\log\left(2 - 2\cos\frac{2\pi m}{n}\right)\right)\\
P_{i,i+n-2} &= \frac{1}{an}\left(\frac{\pi}{2}\left(\cot\frac{\pi(n-1)}{n} - \cot\frac{4\pi}{n} + \cot\frac{3\pi}{n}\right) + \log n\right.\\
&\quad \left. + \frac{1}{2}\sum_{m=1}^{n-1}\left(-\cos\frac{2\pi m(n-1)}{n} + \cos\frac{8\pi m}{n} - \cos\frac{6\pi m}{n}\right)\log\left(2 - 2\cos\frac{2\pi m}{n}\right)\right)\\
P_{i,i+d} &= \frac{1}{an}\left(\frac{\pi}{2}\left(-\cot\frac{\pi(d+2)}{n} + \cot\frac{\pi(d+1)}{n} + \cot\frac{\pi(d-2)}{n} - \cot\frac{\pi(d-1)}{n}\right)\right.\\
&\quad + \frac{1}{2}\sum_{m=1}^{n-1}\left(\left(\cos\frac{2\pi m(d+2)}{n} - \cos\frac{2\pi m(d+1)}{n} + \cos\frac{2\pi m(d-2)}{n} - \cos\frac{2\pi m(d-1)}{n}\right)\right.\\
&\qquad \left.\left. \times \log\left(2 - 2\cos\frac{2\pi m}{n}\right)\right)\right).\\
P_{i,i+2} &= \frac{1}{an}\left(\frac{\pi}{2}\left(-\cot\frac{4\pi}{n} + \cot\frac{3\pi}{n} + \cot\frac{\pi(n-1)}{n}\right) + \log n\right.\\
&\quad \left. + \frac{1}{2}\sum_{m=1}^{n-1}\left(\cos\frac{8\pi m}{n} - \cos\frac{6\pi m}{n} - \cos\frac{2\pi m(n-1)}{n}\right)\log\left(2 - 2\cos\frac{2\pi m}{n}\right)\right)\\
P_{i,i+1} &= \frac{1}{an}\left(\frac{\pi}{2}\left(-\cot\frac{3\pi}{n} + \cot\frac{2\pi}{n} + \cot\frac{\pi}{n}\right) - \log n + n\right.\\
&\quad \left. + \frac{1}{2}\sum_{m=1}^{n-1}\left(\left(\cos\frac{6\pi m}{n} - \cos\frac{4\pi m}{n} - \cos\frac{2\pi m}{n}\right)\log\left(2 - 2\cos\frac{2\pi m}{n}\right)\right)\right)\\
P_{i,i} &= \frac{1}{an}\left(\frac{\pi}{2}\left(\cot\frac{\pi}{n} - \cot\frac{2\pi}{n}\right) + \frac{1}{2}\sum_{m=1}^{n-1}\left(\cos\frac{4\pi m}{n} - \cos\frac{2\pi m}{n}\right)\log\left(2 - 2\cos\frac{2\pi m}{n}\right) - \frac{n}{2}\right)\\
&\quad + 1 - \frac{n^2 + 3n + 3}{2a(n+1)(n+2)}
\end{aligned}
$$

Result: Lower bound



Result: Asymptotic behavior

N	$1/L(N,a)$	$\mathcal{A}$	$\mathcal{A}/L(N,a)$
100	7.43823	0.4528	3.1672
500	7.76437	0.39048	3.03183
1000	7.90483	0.37555	2.96866

- $\mathcal{A}$ is an average of a simulation result of a probability that the hunter catches the rabbit.
- We can show that for $\beta = 1$,

$$\lim_{N\to\infty} \left(\frac{1}{c_*\pi} \log N\right) \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{\mathcal{R}_n^{(N)} = 0\right\}\right) = 1.$$

Example 2. Variable tailed random walk

Let

$$P\{x_t = k\} = \begin{cases} \dfrac{1}{a|k|^{\beta+1}} & (k \neq 0) \\ 1 - \dfrac{2}{a}\displaystyle\sum_{l=1}^{\infty} \dfrac{1}{|l|^{\beta+1}} & (k = 0), \end{cases}$$

where $a \geq 2\sum_{l=0}^{\infty} \frac{1}{|l|^{\beta}}$ and $\beta \in (0, 2]$.

In this case, $c_* = \frac{\pi}{2a\Gamma(\beta+1)\sin(\beta\pi/2)}$ and $\varepsilon = \frac{2-\beta}{2}$.

Lower bound

$\beta = 1$, $a = 2.5$ のとき,

$$\sum_{i=0}^{N-1} p_i^{(N)} \le \frac{5}{\pi^2} \log N + 4.65936.$$

Proposition 1 より

$$\left(\frac{5}{\pi^2} \log N + 4.65936 \right)^{-1} \le \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = (y_n \mod N) \right\} \right).$$

Transition matrix

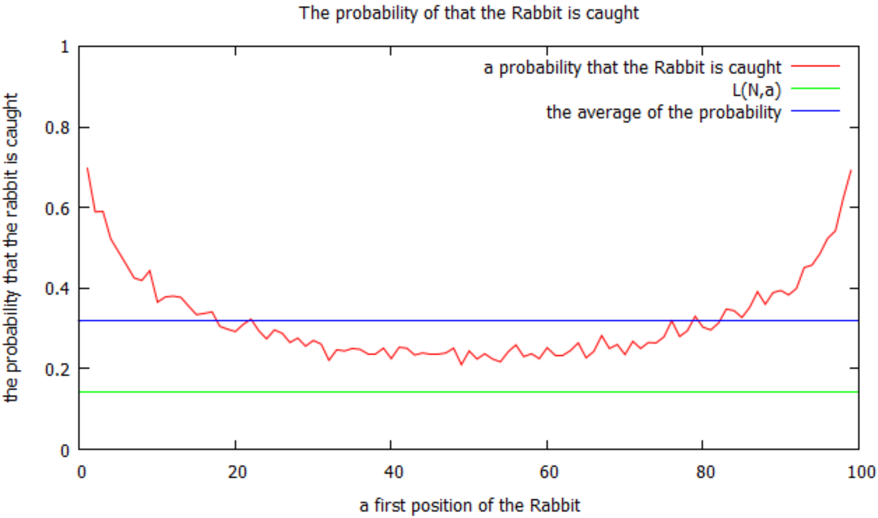
When $\beta = 1$, for $d \neq 0$,

$$\mathbb{P}_{i,i+d} = \frac{1}{aN^2}\left(\psi'\left(\frac{d}{N}\right) + \psi'\left(1 - \frac{d}{N}\right)\right).$$

Then, we can show that

$$\mathbb{P}_{i,i+d} = \begin{cases} \dfrac{\pi^2}{aN^2 \sin^2\left(\frac{\pi d}{N}\right)} & (d \neq 0), \\ 1 - \dfrac{\pi^2}{3a} + \dfrac{\pi^2}{3aN^2} & (d = 0). \end{cases}$$

Result: Lower bound



Result: $N = 100, 500, 1000$

N	$1/L(N,a)$	$\mathcal{A}$	$\mathcal{A}/L(N,a)$
100	6.99237	0.318	2.22357
500	7.80772	0.25924	2.02407
1000	8.15887	0.24015	1.95935

$$\lim_{N\to\infty}\left(\frac{1}{c_*\pi}\log N\right)\mathbb{P}^{(N)}_{\mathcal{R}}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}^{(N)}_n = 0\right\}\right) = 1.$$

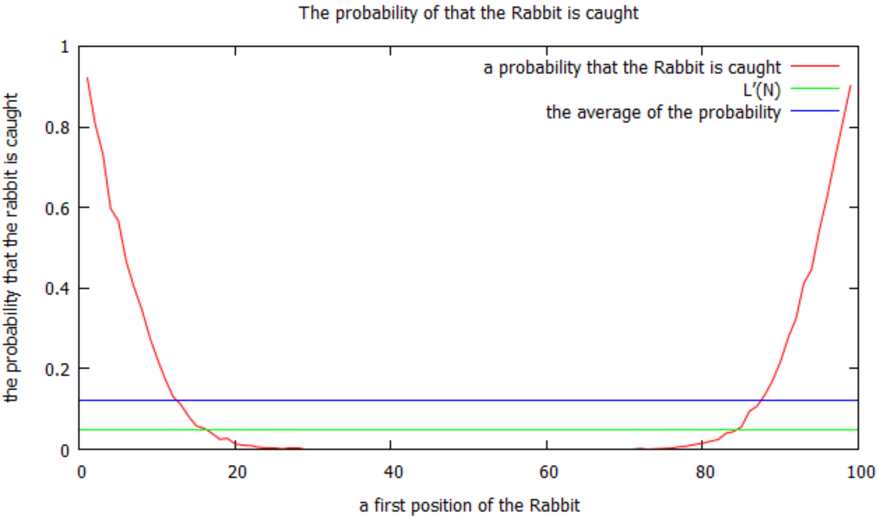
Example 3. We put

$$P\{X_t = k\} = \begin{cases} \dfrac{1}{3} & k \in \{-1, 0, 1\} \\ 0 & k \notin \{-1, 0, 1\}. \end{cases}$$

In this case, $\beta = 2$, $c_* = \frac{1}{3}$ and $\varepsilon = 2$. Proposition 1 より

$$\left(\left(1 + \frac{6}{\pi^2}\right) N^{1/2} + 4.26301\right)^{-1} \leq \mathbb{P}_{\mathcal{R}}^{(N)}\left(\bigcup_{n=1}^{N}\left\{\mathcal{R}_n^{(N)} = (y_n \mod N)\right\}\right).$$

Result: Lower bound



6. Conclusion

Conclusion (1)

- We formalized the Hunter vs Rabbit game using the random walk framework.
- We generalize a probability distribution of the rabbit's strategy using three assumptions.
- We have the general lower bound formula of a probability that the hunter catches the rabbit.

Conclusion (2)

1. If $\beta \in (0, 1)$, the lower bound of a probability is a constant.
2. If $\beta = 1$, the lower bound of a probability is $(\frac{1}{c_* \pi} \log N + c_2)^{-1}$.
3. If $\beta \in (1, 2]$, the lower bound of a probability is $c_4 N^{(1-\beta)/\beta}$.
4. For $\beta = 1$,

$$\lim_{N \to \infty} \left(\frac{1}{c_* \pi} \log N \right) \mathbb{P}_{\mathcal{R}}^{(N)} \left(\bigcup_{n=1}^{N} \left\{ \mathcal{R}_n^{(N)} = 0 \right\} \right) = 1$$

5. We develop a simulation program to simulate the Hunter vs Rabbit game using C++.

Conclusion (3)

About our simulation

- To simulate the Hunter vs Rabbit game in a cycle graph, we have to use the digamma function.
- We can confirm our bounds formulas and asymptotic behavior of those bounds by the result of simulations.

Another

- For Heavy-tailed random walk, we find the parameter that minimize an expected value of time until the hunter catches the rabbit.

G. E. Andrews et al. showed that for $0 < p < q$ and $x \neq 0, -1, -2, \ldots,$

$$\begin{aligned}\psi\left(\frac{p}{q}\right) &= -\gamma - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) - \log q \\ &\quad + \frac{1}{2}\sum_{n=1}^{q-1}\cos\left(\frac{2\pi np}{q}\right)\log\left(2 - 2\cos\left(\frac{2\pi n}{q}\right)\right)\end{aligned}$$

and

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right).$$

G. E. Andrews et al. showed that

$$\psi(1-x)-\psi(x)=\pi\cot(\pi x)=\pi\frac{\cos(\pi x)}{\sin(\pi x)}$$

and

$$\frac{d}{dx}\psi(x)=\sum_{k=0}^{\infty}\frac{1}{(k+x)^2}.$$

RANDOM_SPEED_STRATEGY:

$$\mathcal{R}_t = \lfloor \mathcal{R}_0 + speed * t \rfloor \quad (speed \in [0, 1])$$

HEAVY_TAILED_RANDOM_WALK:

$$\mathcal{R}_t = \begin{cases} \frac{1}{2*sp*(k+1)(k+2)} & (k \neq 0) \\ 1 - \frac{1}{2a} & (k = 0) \end{cases}$$