Algebraic Formalisations of Fuzzy Relations and Their Representation Theorems

by

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Abstract

The aim of this thesis is to develop the *fuzzy relational calculus*. To develop this calculus, we study four algebraic formalisations of fuzzy relations which are called fuzzy relation algebras, Zadeh categories, relation algebras and Dedekind categories, and we strive to arrive at their representation theorems.

The calculus of relations has been investigated since the middle of the nineteenth century. The modern algebraic study of (binary) relations, namely relational calculus, was begun by Tarski. The categorical approach to relational calculus was initiated by Mac Lane and Puppe, and Dedekind categories were introduced by Olivier and Serrato. The representation problem for Boolean relation algebras was proposed by Tarski as the question whether every Boolean relation algebra is isomorphic to an algebra of ordinary homogeneous relations. There are many sufficient conditions that guarantee representability for Boolean relation algebras. Schmidt and Ströhlein gave a simple proof of the representation theorem by using a notion of *point relations*. This notion is a relational (algebraic) characterisation of elements in a set. They also showed the important rôle of point relations in computer science. Following the spirit of Schmidt and Ströhlein, we prove representation theorems for algebraic formalisations of fuzzy relations in this thesis.

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Chapter 1

Introduction

The importance of relations is almost self-evident. Science is, in a sense, the discovery of relations between observables. Zadeh has shown the study of relations to be equivalent to the general study of systems (a system is a relation between an input space and an output space).

Goguen, J.A. [Gog67]

The calculus of relations has been investigated since the middle of the nineteenth century. The modern algebraic study of (binary) relations, namely relational calculus, was begun by Tarski [Tar41]. A definition of Boolean relation algebras first appeared in [JT48], and another equivalent definition appeared in [JT52]. Jónsson and Tarski studied Boolean algebras with operators [JT51, JT52], which is a very general study including Boolean relation algebras. (See [Mad91a] for more details of the history of the study of Boolean relation algebras.) Mac Lane [Mac61] and Puppe [Pup62] exposed a categorical basis for calculus of additive relations. Freyd and Scedrov [FS90] developed and summarized categorical relational calculus, which they called allegories. And various definitions of relation categories exist; for example, Dedekind categories [KFM96, KF97, OS80, OS95] and Schröder categories [Jon88, OS80] were introduced by Olivier and Serrato. (Olivier and Serrato indicated some points of difference and similarity between the two categories in [OS80, OS95].)

In relational calculus one calculates with relations in an element-free style, which makes relational calculus a very useful framework for the study of mathematics, logics and theoretical computer science and also a useful tool for applications. For example nonclassical logics can be formalised by using relation algebras [Orl88a, Orl88b, Orl94]. Furthermore, the theory of nonclassical logics is intensively used in program logics. So relation algebras are fundamental to programs. In fact much of the theory of relation algebras was applied to program semantics and program development [BKSS91, BS91, BZ86, BM97, DOR94, DO96, HH86a, HH86b, HH87, Hut93, KM92, SS93]. The relational calculus can be applied to not only the theory of programs but also databases [BDJM93, BBG$^+$94, JOB94, OJB94], natural languages [Bot92a, Bot92b, Sup76, Sup79, Sup81], graphs [KM94, Miz93, MiK95, SS93], set theory [Kaw95, TG87] and so on. Orlowska described benefit from applying relational calculus to all sorts of scientific research areas in just one sentence in chapter 6 of [BKS97] so clearly:

The main advantages of the relational formalisation are uniformity and modularity.

Actually, once problems in these fields are formalised in terms of relational calculus, these problems can be considered by using formulae of relations, that is, we need only calculus of relations in order to solve the problems. This makes investigation on these fields easier.

Mathematical treatment of uncertain or incomplete information is an important problem in fields of computer science such as expert systems, natural language processing, knowledge representation, pattern recognition, and databases. To treat such informations mathematically, Zadeh [Zad65] introduced the concept of fuzzy sets, and Pawlak [Paw82] introduced the concept of rough sets. It was shown by Cattaneo [Cat96] that the notions of fuzziness and roughness can be formally analyzed in the structure of quasi Brouwer-Zadeh poset. In this thesis we aim at the notion of fuzzy relations. Since Zadeh's invention [Zad65], the concept of fuzzy sets has been extensively investigated in mathematics, science and engineering. Di Nola and Gerla [DG86] studied

nonstandard fuzzy sets. Goguen [Gog67] generalized the concepts of fuzzy sets and relations to taking values on partially ordered sets. Fuzzy relations and calculus of fuzzy relations are very important notions in the theory of fuzzy sets. It is well-known that the notion of fuzzy relations is basic in processing fuzzy information. For example, each rule in a fuzzy expert system is formalised mathematically as a fuzzy relation between the sets describing antecedent and consequent; see 4.1 of [Ped96] written by Kandel et al. for details. Fuzzy relational equations were initiated by Sanchez [San76] and applied to medical models of diagnosis. Later the equations were applied to models in system analysis [Ped85a, Ped85b, Ped87, Ped90a], pattern recognition [Ped90b] and so on; see [Ped96] for more information of fuzzy modelling. On fuzzy relational equations as a methodology for processing fuzzy information in relational structure are summarized by Pedrycts in [Ped91]. In the context of applications of the notion of fuzzy relations to computer science, the notion has been studied [DK93, Nem86]. In the studies, treatment of fuzzy relations were in an element-wise style. On the other hand, as we discussed above, in relational calculus we calculate with relations in an element-free style, which has given us benefits to apply theories of relations to computer science. This fact suggests that if we have fuzzy relational calculus in which we calculate with fuzzy relations in an element-free style, we get benefit as in the case of ordinary relational calculus. This is an original motivation of our research. In fact, fuzzy relation algebras [KF95], which are introduced in chapter 3, gave a theoretical basis to research of fuzzy difunctional dependency in fuzzy relational databases [OJ96], and a rewriting system of fuzzy graphs by using single pushouts [MoK97] based on a study of Zadeh categories [KFM96] which are introduced in chapter 5.

The aim of our research is to develop *fuzzy* relational calculus following Tarski's work. Also we consider representation theorems for fuzzy relation algebras [KF95], Dedekind categories [KFM96, KF97, OS80, OS95], Zadeh categories [KFM96] and relation algebras [Fur97b]. The representation problem for Boolean relation algebras was proposed by Tarski as the question of whether every Boolean relation algebra is isomorphic to an algebra of ordinary homogeneous relations in [Tar41]. This problem for Boolean algebras was solved by Stone [Sto36]. The first negative answer to Boolean relation algebras was given by Lyndon [Lyn50]. Natural constructions for nonrepresentable Boolean relation algebras was provided by Jónsson [Jon59] and Lyndon [Lyn61]. But there are many sufficient conditions that guarantee representability for Boolean relation algebras. The first result was provided by Jónsson and Tarski [JT52] by using a notion of univalent (functional) relations. Maddux and Tarski [MT76] generalized the result in [JT52]. By using a notion of conjugated quasiprojections [TG87], Tarski [Tar53] provided such a condition. All above results were generalized by Maddux [Mad78]. Maddux [Mad91b] also gave a representation result by using a notion of pair-dense. Concerning algebras generated by symmetric and transitive relations, Jónsson [Jon88] and Givant [Giv94] each provided such conditions. (The readers refer the list of historical results of such a problem for Boolean relation algebras, which we introduced above, in chapter 2 of [BKS97].) Schmidt and Ströhlein [SS85] simplified the proof of the representation theorem by using a notion of *point relations*. They also showed the important rôle of point relations in computer science [SS93]. Since it is comfortable for the author to use the notion of point relations concerning application of our relational calculus to computer science and the notion is suitable as a relational (algebraic) characterisation of elements in a set to the author's intuition, we follow the spirit of Schmidt and Ströhlein in this thesis.

This thesis organized as follows:

In chapter 2, we prepare some notions that are necessary in the following chapters, which are definitions of fuzzy relations [Zad65], L-relations [Gog67] and Dedekind categories [OS80, OS95, KFM96, KF97], and their basic properties.

In chapter 3, we provide an algebraic formalisation for fuzzy relations. Fuzzy relations treated here are homogeneous ones on a fixed set X with values in the unit interval $[0,1]$, that is, functions $R : X \times X \to [0,1]$. The set of all such fuzzy relations on X constitutes a fuzzy relation algebra. Unlike Boolean relation algebras, fuzzy relation algebras are not Boolean and so the Schröder rule [SS85], an important axiom of Boolean relation algebras, does not work in our case. Instead of the Schröder rule we prefer to adopt the Dedekind formula. As fuzzy relations are naturally equipped with semi-scalar multiplication by scalars in the unit interval, it

is necessary to add axioms F1 and F3 to the axioms of relation algebras. The axiom F1 requires fuzzy relation algebras to be fuzzy algebras. The notion of fuzzy algebras is a generalisation of algebras formed by functions with values in the unit interval $[0,1]$, that is, functions $a : X \to [0,1]$. Hence a fuzzy algebra is a complete distributive lattice with semi-scalar multiplication by scalars in the unit interval. In the fuzzy theory, we call ordinary relations which are fuzzy relations with values only in the set $\{0,1\}$ *crisp* relations [Gog67]. Thus crisp relations are special ones in fuzzy relations and form a Boolean subalgebra. The concept of crisp relations is well formalised in terms of semi-scalar multiplication, and then we will introduce axiom A3 for semi-Boolean algebras, which claims that crisp relations have complements. The final axiom F4, called point axiom, asserts that pair relations formed by point relations play the rôle of atoms. These are main the differences to the Boolean case. The point axiom F4 enables us to construct a function assigning a concrete fuzzy relation on the set of point relations to an abstract relation in a fuzzy relation algebra. Finally we show a representation, an insertion and an isomorphism theorems for fuzzy relation algebras without assuming the Tarski rule [Tar41, SS85].

In chapter 4, we give an algebraic characterisation of cartesian products of fuzzy relations. Jónsson and Tarski [JT52] characterized cartesian products of Boolean relation algebras by using a notion of ideal relations. Ideal relations are universal with respect to the composition with the universal relation ∇. A representable Boolean relation algebra which is isomorphic to the algebra of all ordinary homogeneous relations has no ideal relation except for the zero relation O and the universal relation ∇. In order to characterize cartesian products of fuzzy relations algebraically, we have to add two axioms for fuzzy relation algebras. The first is called the axiom of cartesian products, which corresponds to the algebraic characterisation of cartesian products of Boolean relation algebras introduced by Jónsson and Tarski [JT52], and the second is called the point axiom for cartesian products. Ideal relations in fuzzy relation algebras are required to be crisp in order to keep general properties the same as properties of such relations in Boolean relation algebras. The second one is defined by using an improved notion of point relations and the point axiom provided in section 3.3, since such point relations and the point axiom don't work on cartesian products of fuzzy relation algebras.

In chapter 5 we provide a categorical formalisation for fuzzy relations from a set X to a set Y. Fuzzy relations are set-functions with values in the unit interval $[0,1]$, that is, functions $R : X \times Y \to [0,1]$. The set of all such fuzzy relations from X to Y constitutes a fuzzy relation category. The category of fuzzy relations is naturally equipped with semi-scalar multiplication by scalars in the unit interval. Also, unlike Boolean relation categories, fuzzy relation categories are not Schröder categories [OS80, Jon88]. Instead of Schröder categories we need to adopt Dedekind categories, named by Olivier and Serrato [OS80, OS95], in order to study fuzzy relation categories. First we show a simple representation theorem for Dedekind categories with a unit object satisfying a condition, called the strict point axiom. Secondly we first state the definition of Zadeh categories as a categorical structure formed by fuzzy relations with sup-min composition. Then some useful properties on point relations, due to Schmidt and Ströhlein [SS85], and the point axiom will be introduced to prove a representation theorem for morphisms in Zadeh categories. In particular, the point axiom enables us to construct a function assigning a concrete fuzzy relation between the sets of all point relations to an abstract relation in Zadeh categories. Finally we show a representation theorem for Zadeh categories satisfying a point axiom without the assumption of existence of unit objects. And it is proved that the representation function is a bijection preserving all operations of Zadeh categories. Therefore the representation function turns out to be a functor. However this functor is not generally an isomorphism of categories since it maps all objects with no points to the empty set. And also we prove that the representation theorem coincides with the simple representation theorem in section 5.1, when the Zadeh category has a unit object satisfying the strict point axiom.

In chapter 6, we consider relation algebras, which may not be Boolean, and show their representation theorem. Relation algebras in the sense of that chapter are equivalent to Dedekind categories [OS80, OS95] (or allegories [FS90]) with just one object. A representation theorem for Dedekind categories is proved in chapter 5 with the assumption of existence of a unit object. The discussion has not assumed the existence of a unit

object, and L-relations are introduced as homogeneous relations on a set X with values on a fixed complete distributive lattice L, that is, functions $R : X \times X \to L$. This study is the first step to prove a representation theorem for Dedekind categories without unit objects. To prove a representation theorem for relation algebras, we use concepts of scalar relations and point relations. The concept of scalar relations is an original one, which is defined as a relation included in the identity relation and which commutes with the greatest relation with respect to composition. In the case of L-relations, scalar relations can be represented as scalar matrices, as easily understood. We use the concept of scalar relations to define a new concept of crisp relations different from that in chapter 3. Also the set of all scalar relations is a complete distributive lattice, which is a sublattice of the relation algebra, and scalar relations represent membership values. Using our concepts of scalar relations and point relations, we define a strict point axiom for relation algebras, and then we prove our representation theorem.

In chapter 7, we study Dedekind categories [OS80, OS95]. One of the aims in that chapter is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness had been introduced in chapter 5 under the assumption that Dedekind categories have unit objects which are an abstraction of singleton (or one-point) sets. To capture the notion of crispness without such an assumption, we use a generalized notion of scalar relations form chapter 6. We first define a preoder among objects of Dedekind categories which compares the lattice structures on objects in a sense. Next we study notions of scalars and crispness for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. Also the scalars are equivalent to the ideals which were introduced by Jónsson and Tarski [JT52]. Then we recall the definition of heterogeneous L-relations, and illustrate a few relationships between crispness and lattice structures of scalars. The other aim of the chapter is to prove another representation theorem for Dedekind categories. Such a theorem for Dedekind categories with a unit object satisfying strict point axiom is also proved in chapter 5. In this chapter we show a representation theorem for uniform Dedekind categories satisfying the strict point axiom without the assumption of existence of unit objects, and it is proved that the representation function is a bijection preserving all operations of Dedekind categories.

Chapter 8 contains a conclusion and outlook into possible future future research.

Chapter 2

Preliminaries

This chapter gives notions and notational conventions of fuzzy relations, L-relations, and Dedekind categories which are needed in this thesis. More precise information on these concepts can be obtained from [Zad65, KF95, Gog67, Fur97b, OS80, OS95, KF95, FS90].

In section 2.1, we introduce the definition of fuzzy relations [Zad65] and their basic properties [Zad65, KF95, Gog67]. In section 2.2, we provide the definition of L-relations [Gog67] and their basic properties [Gog67, Fur97b]. In section 2.3, we give a definition of Dedekind categories [OS80, OS95], a definition of relation algebras [Fur97b], and their basic properties [OS80, OS95, KF95, FS90, Fur97b].

2.1 Fuzzy Relations

Let $[0,1]$ be the unit interval, that is, the set of all real numbers k with $0 \leq k \leq 1$. In this thesis, a real number $k \in [0,1]$ will be called a scalar and denoted by $k, k', k'' \cdots$ and so on. (All real numbers k appearing in the thesis are scalars, that is, $0 \leq k \leq 1$.) The supremum (least upper bound) and the infimum (greatest lower bound) of a family $\{k_\lambda\}_\lambda$ of scalars will be denoted by $\vee_\lambda k_\lambda$ and $\wedge_\lambda k_\lambda$, respectively. In particular, $k \vee k' = \max\{k, k'\}$ and $k \wedge k' = \min\{k, k'\}$. For two elements $k, k' \in [0,1]$ the relative pseudo-complement of k relative to k' will be written as $k \Rightarrow k'$ and defined by

$$k \Rightarrow k' = \begin{cases} 1 & \text{if } k \leq k' \\ k' & \text{otherwise} \end{cases} .$$

Now recall some fundamentals on fuzzy relations [Zad65].

Definition 2.1.1 Given possibly different sets X and Y, a **(heterogeneous) fuzzy relation** R from X to Y, written $R : X \rightharpoonup Y$, is a function $R : X \times Y \to [0,1]$. □

For $(x,y) \in X \times Y$ the value $R(x,y) \in [0,1]$ means the degree to which x and y are related under R. The set of all (heterogeneous) fuzzy relations from X to Y will be denoted by $F\text{-}\mathbf{Rel}(X,Y)$. A fuzzy relation R is contained in a fuzzy relation S, written $R \subseteq S$, if

$$R(x,y) \leq S(x,y)$$

for all $(x,y) \in X \times Y$. The *zero* (empty) relation 0_{XY} and the *universal* relation 1_{XY} are fuzzy relations with

$$0_{XY}(x,y) = 0$$

and

$$1_{XY}(x,y) = 1$$

for all $(x, y) \in X \times Y$, respectively. It is trivial that $\subseteq$ is a complete distributive lattice, and $0_{XY} \subseteq R \subseteq 1_{XY}$ for all fuzzy relations $R : X \multimap Y$. We denote the least upper bound and the greatest lower bound of the family $\{R_\lambda : X \multimap Y\}_\lambda$ of fuzzy relations by $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$, respectively. We then have:

$$(\cup_\lambda R_\lambda)(x, y) = \vee_\lambda R_\lambda(x, y)$$

and

$$(\cap_\lambda R_\lambda)(x, y) = \wedge_\lambda R_\lambda(x, y)$$

for all $(x, y) \in X \times Y$. The composite $RS(= S \circ R) : X \multimap Z$ of a fuzzy relation $R : X \multimap Y$ followed by a fuzzy relation $S : Y \multimap Z$ is defined by

$$(RS)(x, z) = \vee_{y \in Y}(R(x, y) \wedge S(y, z))$$

for all $(x, z) \in X \times Z$. This composition of fuzzy relations is called sup-min composition. The associativity $(RS)T = R(ST)$ holds for all fuzzy relations R, S, and T. The identity relation E_X of a set X is a fuzzy relation such that

$$E_X(x, x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise .} \end{cases}$$

The unit laws $E_X R = R$ and $RE_Y = R$ hold for all $R : X \multimap Y$. The converse (or transpose) $R^\cup : Y \multimap X$ of a fuzzy relation $R : X \multimap Y$ is defined by

$$R^\cup(y, x) = R(x, y)$$

for all $(y, x) \in Y \times X$. The semi-scalar multiplication $kR(= k * R) : X \multimap Y$ of a fuzzy relations $R : X \multimap Y$ by a scalar k is a fuzzy relation such that

$$(kR)(x, y) = kR(x, y)$$

for all $(x, y) \in X \times Y$. For fuzzy relations $S : Y \multimap Z$ and $T : X \multimap Z$, the residue $T \div S : X \multimap Y$ is defined by

$$(T \div S)(x, y) = \vee_{z \in Z}(S(y, z) \Rightarrow T(x, z))$$

for all $(x, y) \in X \times Y$. A fuzzy relation $R : X \multimap Y$ is called *nonzero* if $R \neq 0_{XY}$. A fuzzy relation $R : X \multimap Y$ is called **crisp** [Gog67] if $R(x, y) = 0$ or $R(x, y) = 1$ for all $(x, y) \in X \times Y$. It is now obvious [Gog67] that fuzzy relations together with operations defined above satisfy almost all axioms stated in section 2.3, 5.1 (3.2), and 3.1 of this thesis. Maybe only D3 (R3) (Dedekind formula), D4 (R4) (Residues), Z2(b) (F3(b)), and A3 (semi-Boolean algebra) are not obvious; they will be proved in the following:

Proposition 2.1.1 *Let $R, R' : X \multimap Y, S : Y \multimap Z$, and $T : X \multimap Z$ be fuzzy relations. Then the following holds:*

(a) $RS \cap T \subseteq R(S \cap R^\cup T)$ *(Dedekind formula).*

(b) $RS \subseteq T$ *if and only if* $R \subseteq T \div S$ *(Residues).*

(c) $(kR)S = (kR)(S \cap k1_{XZ})$.

(d) R *is crisp if and only if there is a fuzzy relation* R' *such that* $1_{XY} = R \cup R'$ *and* $R \cap R' = 0_{XY}$.

(e) R *is crisp if and only if* $R \cap k1_{XY} = kR$ *for all scalars* k.

Proof. (a) With $R^\cup(y, x) \wedge T(x, z) \leq (R^\cup T)(y, z)$, the Dedekind formula follows from

$$\begin{array}{rcl} (RS \cap T)(x, z) & = & \vee_{y \in Y}(R(x, y) \wedge S(y, z)) \wedge T(x, z) \\ & = & \vee_{y \in Y}(R(x, y) \wedge S(y, z) \wedge T(x, z)) \\ & = & \vee_{y \in Y}(R(x, y) \wedge S(y, z) \wedge R^\cup(y, x) \wedge T(x, z)) \\ & \leq & \vee_{y \in Y}(R(x, y) \wedge S(y, z) \wedge (R^\cup T)(y, z)) \\ & = & \vee_{y \in Y}[R(x, y) \wedge (S \cap R^\cup T)(y, z)] \\ & = & [R(S \cap R^\cup T)](x, z) \end{array}$$

for all $(x,z) \in X \times Z$.
(b) follows from the following equivalence:

$$\begin{array}{lll} RS \subseteq T & \iff & \forall x \forall z : (RS)(x,z) \leq T(x,z) \\ & \iff & \forall x \forall z : \vee_{y \in Y}(R(x,y) \wedge S(y,z)) \leq T(x,z) \\ & \iff & \forall x \forall z \forall y : R(x,y) \wedge S(y,z) \leq T(x,z) \\ & \iff & \forall x \forall z \forall y : R(x,y) \leq S(y,z) \Rightarrow T(x,z) \\ & \iff & \forall x \forall y : R(x,y) \leq \wedge_{z \in Z}(S(y,z) \Rightarrow T(x,z)) \\ & \iff & \forall x \forall y : R(x,y) \leq (T \div S)(x,y) \\ & \iff & R \subseteq T \div S \ . \end{array}$$

(c) is immediate from

$$\begin{array}{lll} [(kR)(S \cap k1_{YZ})](x,z) & = & \vee_{y \in Y}(kR(x,y) \wedge S(y,z) \wedge k) \\ & = & \vee_{y \in Y}(kR(x,y) \wedge S(y,z)) \\ & = & [(kR)S](x,z) \end{array}$$

for all $(x,z) \in X \times Z$.
(d) Assume that R is crisp, and define $R'(x,y) = 1 - R(x,y)$ for all $(x,y) \in X \times Y$. Then it is clear that $R \cup R' = 1_{XY}$ and $R \cap R' = 0_{XY}$. The converse is trivial (Cf. Proof of Proposition 3.1.2(b)).
(e) It is immediate from

$$\begin{array}{lll} R \cap k1_{XY} = kR & \iff & \forall k \in [0,1] : R(x,y) \wedge k = kR(x,y) \\ & \iff & R(x,y) = 0 \text{ or } R(x,y) = 1 \end{array}$$

for all $(x,y) \in X \times Y$. □

The formula (a) in the last proposition is called "modular law" in [FS90, BM97]; in [Rig48, SS85, SS93] the Dedekind rule is given as $QR \cap S \subseteq (Q \cap SR^{\cup})(R \cap Q^{\cup}S)$. Since these formulae are equivalent, we use the name "Dedekind formula" for the shorter variant in this thesis.

In relational calculus ([FS90, Kaw95, SS93]) a function $R : X \rightharpoonup Y$ is a univalent

$$R^{\cup}R \subseteq E_Y$$

and total

$$E_X \subseteq RR^{\cup}$$

relation. Note that

$$\begin{array}{lll} R^{\cup}R \subseteq E_Y & \iff & \forall y,y' : R^{\cup}R(y,y') = \vee_{x \in X}(R^{\cup}(y,x) \wedge R(x,y')) \leq E_Y(y,y') \\ & \iff & \forall x,y,y' : y \neq y' \Longrightarrow R(x,y) \wedge R(x,y') = 0 \ , \end{array}$$

and

$$\begin{array}{lll} E_X \subseteq RR^{\cup} & \iff & \forall x : R^{\cup}R(x,x) = \vee_{y \in X}(R(x,y) \wedge R^{\cup}(y,x)) = 1 \\ & \iff & \forall x : \vee_{y \in X} R(x,y) = 1 \ . \end{array}$$

Thus total relations are nonzero if neither X nor Y are empty.

To capture the concept of points in Boolean relation algebras Schmidt and Ströhlein [SS85] defined a notion of vector relations. In our setting, a vector relation (left ideal) $R : X \rightharpoonup Y$ is a fuzzy relation such that

$$1_{XX}R = R$$

(in [SS85] such a relation is defined as right ideal, i.e. $R1_{YY} = R$), namely,

$$\begin{array}{lll} 1_{XX}R = R & \iff & \forall x,y : R(x,y) = \vee_{x' \in X}(1_{XX}(x,x') \wedge R(x',y)) \\ & \iff & \forall x,y : R(x,y) = \vee_{x' \in X} R(x',y) \\ & \iff & \forall x,x',y : R(x,y) = R(x',y) \ . \end{array}$$

The following lemma motivates the definition of point relations in fuzzy relation algebras.

Lemma 2.1.1 *Let X be a nonempty set. If $R : X \rightharpoonup Y$ is a crisp fuzzy relation with $R^{\cup}R \subseteq E_Y$, $E_X \subseteq RR^{\cup}$ and $1_{XX}R = R$, then there is an element $y_0 \in Y$ such that for all $(x, y) \in X \times Y$, $R(x, y) = 1$ if $y = y_0$ and $R(x, y) = 0$ otherwise.*

Proof. Since X is not empty, there is an element $x_0 \in X$. As R is crisp and total, there is an element $y_0 \in Y$ such that $R(x_0, y_0) = 1$. But it follows from $1_{XX}R = R$ that $R(x, y_0) = 1$ for all $x \in X$, and from $R^{\cup}R \subseteq E_Y$ that $R(x, y) = R(x, y) \wedge R(x, y_0) = 0$ for $y \neq y_0$. □

Definition 2.1.2 A **(homogeneous) fuzzy relation** R on a fixed set X is a function $R : X \times X \to [0, 1]$.□

The set of all (homogeneous) fuzzy relations on a fixed set X will be denoted by F-**Rel**(X). Every operator of homogeneous fuzzy relations is defined in the same way as in heterogeneous case without attention to domains of relations. Also basic properties we discuss in this section hold in homogeneous case. (See [KF95] for details.) The zero relation, the universal relation and the identity relation in F-**Rel**(X) are denoted by 0_X, 1_X and E_X, respectively.

2.2 L-Relations

Let $L = (L, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$ be a fixed complete distributive lattice (or, a complete Heyting algebra) [Birk48] with least element $\mathbf{0}$ and greatest element $\mathbf{1}$. Elements of the complete distributive lattice L will be denoted by $l, l', l'', \cdots$ and so on. The supremum (least upper bound) and the infimum (greatest lower bound) of a family $\{l_\lambda\}$ of elements in L will be denoted by $\vee_\lambda l_\lambda$ and $\wedge_\lambda l_\lambda$, respectively. For two elements $l, l' \in L$ the relative pseudo-complement of l relative to l' will be written as $l \Rightarrow l'$. Now recall some fundamentals on L-relations [Gog67].

Definition 2.2.1 Given possibly different sets X and Y, a **(heterogeneous) L-relation** R from X to Y, written $R : X \rightharpoonup Y$, is a function $R : X \times Y \to L$. □

The set of all (heterogeneous) L-relations from X into Y will be denoted by L-**Rel**(X, Y). An L-relation R is contained in an L-relation S, written $R \subseteq S$, if

$$R(x, y) \leq S(x, y)$$

for all $(x, y) \in X \times Y$. The *zero* (empty) relation 0_{XY} and the *universal* relation 1_{XY} are L-relations with

$$0_{XY}(x, y) = \mathbf{0}$$

and

$$1_{XY}(x, y) = \mathbf{1}$$

for all $(x, y) \in X \times Y$, respectively. It is trivial that $\subseteq$ is a complete distributive lattice, and $0_{XY} \subseteq R \subseteq 1_{XY}$ for all L-relations R. We denote the least upper bound and the greatest lower bound of a family $\{R_\lambda : X \rightharpoonup Y\}_\lambda$ of L-relations by $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$, respectively. We then have:

$$(\cup_\lambda R_\lambda)(x, y) = \vee_\lambda R_\lambda(x, y)$$

and

$$(\cap_\lambda R_\lambda)(x, y) = \wedge_\lambda R_\lambda(x, y)$$

for all $(x, y) \in X \times Y$. The composite $RS (= S \circ R) : X \rightharpoonup Z$ of an L-relation $R : X \rightharpoonup Y$ followed by an L-relation $S : Y \rightharpoonup Z$ is defined by

$$(RS)(x, z) = \vee_{y \in Y}(R(x, y) \wedge S(y, z))$$

for all $(x,z) \in X \times Z$. This composition of L-relations is called sup-inf composition. The associativity $(RS)T = R(ST)$ holds for all L-relations R, S and T. The identity relation E_X of a set X is an L-relation such that

$$E_X(x,x') = \begin{cases} \mathbf{1} & \text{if } x = x' \\ \mathbf{0} & \text{otherwise .} \end{cases}$$

The unit laws $E_X R = R$ and $RE_Y = R$ hold for all $R : X \rightharpoonup Y$. The converse (or transpose) $R^\cup : Y \rightharpoonup X$ of an L-relation $R : X \rightharpoonup Y$ is defined by

$$R^\cup(y,x) = R(x,y)$$

for all $(y,x) \in Y \times X$. For L-relations $S : Y \rightharpoonup Z$ and $T : X \rightharpoonup Z$, the residue $T \div S : X \rightharpoonup Y$ is defined by

$$(T \div S)(x,y) = \wedge_{z \in Z}(S(y,z) \Rightarrow T(x,z))$$

for all $(x,y) \in X \times Y$. An L-relation $R : X \rightharpoonup Y$ is called *nonzero* if $R \neq O_{XY}$. It is now obvious [Gog67] that L-relations together with operations defined above satisfy all axioms of Dedekind categories stated below in section 2.3. Maybe only D3 (R3) (Dedekind formula) and D4 (R4) (Residues) are not so obvious; they will be proved in the following:

Proposition 2.2.1 *Let $R : X \rightharpoonup Y, S : Y \rightharpoonup Z$ and $T : X \rightharpoonup Z$ be L-relations. Then the following holds:*

(a) *$RS \cap T \subseteq R(S \cap R^\cup T)$ (Dedekind formula).*

(b) *$RS \subseteq T$ if and only if $R \subseteq T \div S$ (Residues).*

Proof. (a) With $R^\cup(y,x) \wedge T(x,z) \leq (R^\cup T)(y,z)$, Dedekind formula follows from

$$\begin{array}{rcl} (RS \cap T)(x,z) & = & \vee_{y \in Y}(R(x,y) \wedge S(y,z)) \wedge T(x,z) \\ & = & \vee_{y \in Y}(R(x,y) \wedge S(y,z) \wedge T(x,z)) \\ & = & \vee_{y \in Y}(R(x,y) \wedge S(y,z) \wedge R^\cup(y,x) \wedge T(x,z)) \\ & \leq & \vee_{y \in Y}(R(x,y) \wedge S(y,z) \wedge (R^\cup T)(y,z)) \\ & = & \vee_{y \in Y}[R(x,y) \wedge (S \cap R^\cup T)(y,z)] \\ & = & [R(S \cap R^\cup T)](x,z) \end{array}$$

for all $(x,z) \in X \times Z$.
(b) follows from the following equivalence:

$$\begin{array}{rcl} RS \subseteq T & \iff & \forall x \forall z : (RS)(x,z) \leq T(x,z) \\ & \iff & \forall x \forall z : \vee_{y \in Y}(R(x,y) \wedge S(y,z)) \leq T(x,z) \\ & \iff & \forall x \forall z \forall y : R(x,y) \wedge S(y,z) \leq T(x,z) \\ & \iff & \forall x \forall z \forall y : R(x,y) \leq S(y,z) \Rightarrow T(x,z) \\ & \iff & \forall x \forall y : R(x,y) \leq \wedge_{z \in Z}(S(y,z) \Rightarrow T(x,z)) \\ & \iff & \forall x \forall y : R(x,y) \leq (T \div S)(x,y) \\ & \iff & R \subseteq T \div S \ . \end{array}$$

□

In this thesis we will call a homogeneous L-relation R on a set X **scalar**, if there is an $l \in L$ such

$$R(x,x') = \begin{cases} l & \text{if } x = x' \\ \mathbf{0} & \text{otherwise .} \end{cases}$$

Now we denote the set of all such scalar L-relations of a set X $\mathcal{S}(X)$. Then it is clear that the tuple $(\mathcal{S}(X), \subseteq, \cup, \cap, 0_X, E_X)$ is a complete distributive lattice with least element 0_X and greatest element E_X, and also that is isomorphic to L. Moreover, scalar L-relations can be characterized algebraically:

Proposition 2.2.2 *R is a scalar L-relation of a set X if and only if $R \subseteq E_X$ and $R1_{XX} = 1_{XX}R$.*

Proof. Remark that $R1_{XX}(x,y) = \vee_{z\in X}(R(x,z)\wedge 1_{XX}(z,y)) = R(x,x)\wedge 1_{XX}(x,y) = R(x,x)$ for all $x,y \in X$ if $R \subseteq E_X$. (Similarly $1_{XX}R(x,y) = 1_{XX}(x,y) \wedge R(y,y)$.) Now assume that R is a scalar L-relation. Then it is trivial that $R \subseteq E_X$. Thus $R1_{XX}(x,y) = R(x,x) \wedge 1_{XX}(x,y) = 1_{XX}(x,y) \wedge R(y,y) = 1_{XX}R(x,y)$. Next assume that $R \subseteq E_X$ and $R1_{XX} = 1_{XX}R$. By $R \subseteq E_X$ we obtain $R(x,y) = \mathbf{0}$ if $x \neq y$. Also $R(x,x) = R1_{XX}(x,y) = 1_{XX}R(x,y) = R(y,y)$ for all $x,y \in X$. Therefore R is a scalar L-relation. □

Definition 2.2.2 A **(homogeneous) L-relation** R on a fixed set X is a function $R : X \times X \to L$. □

The set of all (homogeneous) L-relations on a fixed set X will be denoted by L-**Rel**(X). Every operator of homogeneous L-relations is defined in the same way as in the heterogeneous case without attention to domains of L-relations. Also basic properties we discuss in this section hold in homogeneous case. (See [Fur97b] for details.) The zero relation, the universal relation and the identity relation in L-**Rel**(X) are denoted by 0_X, 1_X and E_X, respectively.

2.3 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call Dedekind categories following Olivier and Serrato [OS80, OS95]. Dedekind categories are equivalent to division allegories introduced by Freyd and Scedrov [FS90].

Throughout this thesis, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoonup Y$, and the composite of a morphism $\alpha : X \rightharpoonup Y$ followed by a morphism $\beta : Y \rightharpoonup Z$ will be written as $\alpha\beta : X \rightharpoonup Z$. We denote the identity morphism on an object X by id_X.

Definition 2.3.1 A **Dedekind category** $\mathcal{D}$ is a category satisfying the following:
D1. [**Complete Distributive Lattice**] For all pairs of objects X and Y the hom-set $\mathcal{D}(X,Y)$ consisting of all morphisms of X into Y is a complete distributive lattice with the least morphism O_{XY} and the greatest morphism ∇_{XY}.
D2. [**Involution**] There is given a monotone and involutive contravariant functor $^\sharp : \mathcal{D} \to \mathcal{D}$. That is, for all morphisms $\alpha, \alpha' : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$, the following laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
D3. [**Dedekind Formula**] For all morphisms $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ and $\gamma : X \rightharpoonup Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [**Residues**] For all morphisms $\beta : Y \rightharpoonup Z$ and $\gamma : X \rightharpoonup Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoonup Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoonup Y$. □

Note that complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras. It is clear that both of categories of all sets and fuzzy relations without scalar multiplications and of all sets and L-relations are Dedekind categories.

Example 2.3.1 Consider a category Rel_0 whose objects are all nonempty sets and in which a hom-set $Rel_0(X,Y)$ between objects X and Y is the set of all (binary) relations on X if $X = Y$, and $\nabla_{XY} = O_{XY}$ otherwise. That is, a hom-set $Rel_0(X,Y)$ is a singleton set when X and Y are distinct. Then it is easy to verify that the category Rel_0 is a Dedekind category. The conditions (D1) and (D2) are trivial, and (D3) and (D4) also hold as follows: If $X = Y = Z$, then (D3) and (D4) are clear. If $X = Y \neq Z$, then $\beta = O_{YZ}$, $\gamma = O_{XZ}$ and $\gamma \div \beta = \nabla_{XX}$. If $X \neq Y$, then $\alpha = O_{XY}$ and $\gamma \div \beta = O_{XY}$. □

Throughout the rest of this section, all discussions will assume a fixed Dedekind category $\mathcal{D}$. The greatest morphism ∇_{XY} is called the *universal* morphism and the least morphism 0_{XY} the *zero* morphism. A morphism

is *nonzero* if it is not equal to the zero morphism. An object X is called *empty* if $\nabla_{XX} = 0_{XX}$, and *nonempty* otherwise.

We introduce some basic properties of Dedekind categories.

Proposition 2.3.1 *Let $\alpha, \alpha' : X \rightharpoonup Y$ and $\beta, \beta' : Y \rightharpoonup Z$ be morphisms in $\mathcal{D}$. Then the following holds:*

(a) $O^\sharp_{XY} = O_{YX}$, $\nabla^\sharp_{XY} = \nabla_{YX}$ *and* $\mathrm{id}^\sharp_X = \mathrm{id}_X$.

(b) $(\sqcup_\lambda \alpha_\lambda)\beta = \sqcup_\lambda \alpha_\lambda \beta$.

(c) *If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$.*

(d) $\nabla_{XX}\nabla_{XY} = \nabla_{XY}\nabla_{YY} = \nabla_{XY}$.

(e) $(\alpha \sqcup \beta)^\sharp = \alpha^\sharp \sqcup \beta^\sharp$ *and* $(\alpha \sqcap \beta)^\sharp = \alpha^\sharp \sqcap \beta^\sharp$.

(f) $O_{XY}\beta = O_{XZ}$ *and* $\alpha O_{YZ} = O_{XZ}$.

Proof. (a) $O^\sharp_{XY} \sqsubseteq (O^\sharp_{YX})^\sharp = O_{YX}$ since $O_{XY} \sqsubseteq O^\sharp_{YX}$, and $\nabla_{YX} = (\nabla^\sharp_{YX})^\sharp \sqsubseteq \nabla^\sharp_{XY}$ since $\nabla^\sharp_{XY} \sqsubseteq \nabla_{YX}$, and $\mathrm{id}^\sharp_X = \mathrm{id}^\sharp_X \mathrm{id}_X = \mathrm{id}^\sharp_X (\mathrm{id}^\sharp_X)^\sharp = (\mathrm{id}^\sharp_X \mathrm{id}_X)^\sharp = (\mathrm{id}^\sharp_X)^\sharp = \mathrm{id}_X$ by axiom D2.
(b) follows from the following equivalence:

$$\begin{array}{lll} (\sqcup_\lambda \alpha_\lambda)\beta \sqsubseteq \gamma & \iff & \sqcup_\lambda \alpha_\lambda \sqsubseteq \gamma \div \beta \\ & \iff & \forall \lambda : \alpha_\lambda \sqsubseteq \gamma \div \beta \\ & \iff & \forall \lambda : \alpha_\lambda \beta \sqsubseteq \gamma \\ & \iff & \sqcup_\lambda \alpha_\lambda \beta \sqsubseteq \gamma \end{array}$$

for each $\gamma : X \rightharpoonup Z$ by axiom D4.
(c) If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta \sqcup \alpha'\beta = (\alpha \sqcup \alpha')\beta = \alpha'\beta \sqsubseteq \alpha'\beta \sqcup \alpha'\beta' = \alpha'(\beta \sqcup \beta') = \alpha'\beta'$ by (b).
(d) It is clear that $\nabla_{XX}\nabla_{XY} \sqsubseteq \nabla_{XY}$ and $\mathrm{id}_X \sqsubseteq \nabla_{XX}$ by axiom D1. So $\nabla_{XX}\nabla_{XY} = \nabla_{XY}$ holds. Similarly it can be shown that $\nabla_{XY}\nabla_{YY} = \nabla_{XY}$.
(e) By axioms D1 and D2 we have $\alpha^\sharp \sqcup \beta^\sharp \sqsubseteq (\alpha \sqcup \beta)^\sharp$. Hence $\alpha \sqcup \beta = \alpha^{\sharp\sharp} \sqcup \beta^{\sharp\sharp} \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^\sharp$ and $(\alpha \sqcup \beta)^\sharp \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^{\sharp\sharp} = \alpha^\sharp \sqcup \beta^\sharp$.
(f) follows from the following equivalences:

$$O_{XY} \sqsubseteq O_{XZ} \div \beta \iff O_{XY}\beta \sqsubseteq O_{XZ} \ ,$$

and also

$$\begin{array}{lll} O_{ZY} \sqsubseteq O_{ZX} \div \alpha^\sharp & \iff & O_{ZY}\alpha^\sharp \sqsubseteq O_{ZX} \\ & \iff & O^\sharp_{YZ}\alpha^\sharp \sqsubseteq O^\sharp_{XZ} \\ & \iff & (\alpha O_{YZ})^\sharp \sqsubseteq O^\sharp_{XZ} \\ & \iff & \alpha O_{YZ} \sqsubseteq O_{XZ} \end{array}$$

by axioms D1, D4, and D2. □

From statement (c) in the last proposition, $(\sqcap_\lambda \alpha_\lambda)\beta \sqsubseteq \sqcap_\lambda \alpha_\lambda$ holds. Also statement (d) in the last proposition indicates that if $\nabla_{XY} \neq O_{XY}$ then both of X and Y are nonempty.

Next we provide the axioms R1-R4 of relation algebras. A relation algebra $\mathcal{R}$, which will be defined below, is an algebraic structure over a nonempty set $\mathcal{R}$ of elements called "relations". Originally, relation algebras were formalised by Tarski [Tar41] as complete *Boolean algebras* with composition and converse. But in this thesis, relation algebras are only complete *distributive lattices* with composition and converse. In other words, relation algebras (in this thesis) are Dedekind categories [OS80, OS95] with just one object.

Definition 2.3.2 A **relation algebra** $\mathcal{R} = (\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, O, \nabla, \mathrm{id})$ is an algebraic structure over a nonempty set $\mathcal{R}$ satisfying the following:
R1. [**Complete Distributive Lattice**] The tuple $(\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, O, \nabla)$ is a complete distributive lattice.
R2. [**Involutive Monoid**] The tuple $(\mathcal{R}, ;, ^\sharp, \mathrm{id}, O)$ is an involutive monoid with unit element id and zero

element O. That is,
(a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, (b) $\alpha\mathrm{id} = \mathrm{id}\alpha = \alpha$, (c) $(\alpha^\sharp)^\sharp = \alpha$, (d) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (e) If $\alpha \sqsubseteq \beta$, then $\alpha^\sharp \sqsubseteq \beta^\sharp$.
R3. [**Dedekind Formula**] $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$.
R4. [**Residues**] For all relations β and γ the residue (or division, weakest precondition) $\gamma \div \beta$ is a relation such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all relations α. □

It is clear that algebras F-**Rel**$(X) = (F\text{-}\mathbf{Rel}(X), \subseteq, \cup, \cap, \circ, ^\cup, 0_X, 1_X, E_X)$ of fuzzy relations and L-**Rel**(X) $= (L\text{-}\mathbf{Rel}(X), \subseteq, \cup, \cap, \circ, ^\cup, 0_X, 1_X, E_X)$ of L-relations on a fixed set X are relation algebras. Let $\mathcal{R} = (\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, O, \nabla, \mathrm{id})$ be a relation algebra. A relation algebra $\mathcal{R}$ with $\nabla = O$ is trivial and not worth mentioning. Throughout the rest of the thesis all discussions will assume a fixed relation algebra $\mathcal{R}$ with $\nabla \neq O$. The basic properties of Dedekind categories we discussed in this section also hold in relation algebras without attention to domains.

Chapter 3

A Representation Theorem for Fuzzy Relation Algebras

The aim of the chapter is to provide an algebraic formalisation for fuzzy relations. Fuzzy relations treated here are homogeneous relations on a fixed set X with values in the unit interval $[0,1]$, that is, functions $R : X \times X \to [0,1]$. The set of all such fuzzy relations on X constitutes a fuzzy relation algebra. Unlike Boolean relation algebras, fuzzy relation algebras are not Boolean and so the Schröder rule [SS85], an important axiom of Boolean relation algebras, does not work in our case. Instead of the Schröder rule we prefer to adopt the Dedekind formula, which is stated as axiom R3 in definition 2.3.2. Since fuzzy relations are naturally equipped with semi-scalar multiplication by scalars in the unit interval, additional axioms F1 and F3 to the axioms of relation algebras are required. However, the set of all crisp [Gog67] relations is contained in the set of all fuzzy relations and forms a Boolean subalgebra. The concept of crisp relations is well formalised in terms of semi-scalar multiplication, and then we introduce axiom A3 for semi-Boolean algebras, which ensures that crisp relations have complements (Cf. Proposition 2.1.1(d)). The final axiom F4, called point axiom, asserts that pair relations formed by point relations play the rôle of atoms. These are main the differences from the Boolean case.

Section 3.1 provides a set of the axioms A1–A3 for fuzzy algebras and the basic properties of fuzzy relation algebras. Section 3.2 provides a set of the axioms F1–F3 for fuzzy relation algebras and the basic properties of fuzzy relation algebras. In section 3.3, we first state the definition of point relations and a point axiom F4. Then some properties of point relations are given that will be useful for the proof of the main results. In particular, the point axiom F4 enables us to construct a function assigning a concrete fuzzy relation on the set of point relations to an abstract relation in a fuzzy relation algebra. In section 3.4, we show a representation, an insertion and an isomorphism theorem for fuzzy relation algebras without assumption of the Tarski rule [Tar41, SS85].

This chapter is based on [KF95].

3.1 Fuzzy Algebras

In this section we first introduce a notion of fuzzy algebras as a mathematical structure formed by fuzzy sets, and describe some properties of fuzzy algebras. In short fuzzy algebras are complete distributive lattices with semi-scalar multiplications and a semi-Boolean property. Throughout the chapter, real numbers $k \in [0,1]$ will be called scalars, where $[0,1]$ is the unit interval, that is, the set of all real numbers k with $0 \leq k \leq 1$.

Definition 3.1.1 A **fuzzy algebra** $\mathcal{A} = (\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \cdot, O, \nabla)$ is an algebraic structure over a nonempty set $\mathcal{A}$ satisfying the following:

A1. [**Complete Distributive Lattice**] The tuple $(\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, O, \nabla)$ is a complete distributive lattice with the least element O and the greatest element ∇.

A2. [**Semi-Scalar Multiplication**] An operation $\cdot : [0,1] \times \mathcal{A} \to \mathcal{A}$ is a semi-scalar multiplication of $\mathcal{A}$. That

is,
(a) $0a = O$ *and* $1a = a$, (b) $k(k'a) = (kk')a$, (c) $k(\sqcup_\lambda a_\lambda) = \sqcup_\lambda ka_\lambda$ *and* $k(\sqcap_\lambda a_\lambda) = \sqcap_\lambda ka_\lambda$, (d) $(\wedge_\lambda k_\lambda)a = \sqcap_\lambda k_\lambda a$, (e) If $ka \sqsubseteq kb$ and $0 < k$, then $a \sqsubseteq b$.
(The semi-scalar multiplication $k \cdot a$ of $a \in \mathcal{A}$ by a scalar $k \in [0,1]$ will be written as ka, unless confusion occurs.)
A3. [**Semi-Boolean Algebra**] If $a \sqcap k\nabla = ka$ for all scalars k, then there is an element b such that $a \sqcup b = \nabla$ and $a \sqcap b = O$. □

It is obvious that each algebra $(F\text{-}\mathbf{Rel}(X), \subseteq, \cup, \cap, *, 0_X, 1_X)$ is a fuzzy algebra. A fuzzy algebra $\mathcal{A} = (\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \cdot, O, \nabla)$ with $\nabla = O$ is trivial and not worth mentioning. Throughout the rest of this section we restrict our attention to fuzzy algebras with $\nabla \neq O$.

Proposition 3.1.1 *Let a, b be elements of a fuzzy algebra $\mathcal{A}$ and k, k' scalars. Then the following holds:*

(a) *If $a \sqsubseteq b$, then $ka \sqsubseteq kb$.*

(b) *If $k \leq k'$, then $ka \sqsubseteq k'a$. In particular, $ka \sqsubseteq a$ and $kO = O$.*

(c) $ka \sqcup k'a = (k \vee k')a$

(d) *If $k\nabla = \nabla$, then $k = 1$.*

(e) *If $k\nabla \sqsubseteq k'\nabla$, then $k \leq k'$.*

Proof. (a) If $a \sqsubseteq b$, then $ka = k(a \sqcap b) = ka \sqcap kb \sqsubseteq kb$ by axiom A2(c).
(b) If $k \leq k'$, then $ka = (k \wedge k')a = ka \sqcap k'a \sqsubseteq k'a$ by axiom A2(d). In particular, $ka \sqsubseteq 1a = a$ by axiom A2(a) and $k \leq 1$, and also $O \sqsubseteq kO \sqsubseteq 1O = O$ by axioms A1 and A2(a).
(c) Assume $k \leq k'$. Then $ka \sqcup k'a = k'a = (k \vee k')a$ by (b).
(d) Assume $k\nabla = \nabla$ and $0 \leq k < 1$. Then by axiom A2(b) it is trivial that $k^n\nabla = \nabla$ for all natural numbers n. Hence it holds that $\nabla = \sqcap_{n \leq 0} k^n\nabla = (\wedge_{n \leq 0} k^n)\nabla = 0\nabla = O$ by axiom A2(d), which contradicts the hypothesis $\nabla \neq O$.
(e) Assume $k\nabla \sqsubseteq k'\nabla$ and $k' < k$. Then $0 \leq k'/k < 1$ and $k\nabla \sqsubseteq k[(k'/k)\nabla]$. Therefore $\nabla \sqsubseteq (k'/k)\nabla$ by axiom A2(e) and so $1 = k'/k$ by (d), which contradicts the assumption $k' < k$. □

Following [KF95] the concept of crisp elements in fuzzy algebras is defined as follows:

Definition 3.1.2 An element a of a fuzzy algebra $\mathcal{A}$ is **crisp** if $a \sqcap k\nabla = ka$ for all scalars k. □

In the above definition of crisp elements of fuzzy algebras it is trivial that $ka \sqsubseteq a \sqcap k\nabla$ by propositions 3.1.1(a) and 3.1.1(b). Note that $\nabla \sqcap k\nabla = k\nabla$ since $k\nabla \sqsubseteq \nabla$ by axiom A1. This means that the universal element ∇ is crisp. Also the zero element O is clearly crisp.

Proposition 3.1.2 *Let a,b be elements of $\mathcal{A}$ and k a scalar. Then the following holds:*

(a) *If a and b are crisp, then so are $a \sqcup b$ and $a \sqcap b$.*

(b) *If $a \sqcup b = \nabla$ and $a \sqcap b = O$, then both of a and b are crisp.*

(c) *If b is crisp and $ka \sqsubseteq b$ for $k > 0$, then $a \sqsubseteq b$.*

(d) *If a and b are crisp and $a \sqsubseteq b$, then $a \sqcap kb = ka$ for each scalar k.*

Proof. (a) Let a and b be crisp. Then $(a \sqcup b) \sqcap k\nabla = (a \sqcap k\nabla) \sqcup (b \sqcap k\nabla) = ka \sqcup kb = k(a \sqcup b)$ by axioms A1 and A2(c). Also $(a \sqcap b) \sqcap k\nabla = (a \sqcap k\nabla) \sqcap (b \sqcap k\nabla) = ka \sqcap kb = k(a \sqcap b)$ by axiom A2(c).
(b) Assume $a \sqcup b = \nabla$ and $a \sqcap b = O$. First note that $ka \sqsubseteq a$ and $a \sqcap kb = O$ by proposition 3.1.1(b). Then $a \sqcap k\nabla = a \sqcap k(a \sqcup b) = (a \sqcap ka) \sqcup (a \sqcap kb) = ka$ by axioms A2(c) and A1.

(c) Note that $ka \sqsubseteq k\nabla$ by proposition 3.1.1(a). As b is crisp we have $ka \sqsubseteq b \sqcap k\nabla = kb$. Hence $a \sqsubseteq b$ by axiom A2(e).
(d) Let a and b be crisp with $a \sqsubseteq b$. Then $a \sqcap kb = a \sqcap (b \sqcap k\nabla) = (a \sqcap k\nabla) \sqcap b = ka$. □

From the last proposition 3.1.2(a) and 3.1.2(b) it is immediate that the set of all crisp elements in a fuzzy algebra $\mathcal{A}$ forms a Boolean algebra.

3.2 Fuzzy Relation Algebras

Section 3 provides a set of the axioms F1-F3 for fuzzy relation algebras, and some basic properties of fuzzy relation algebras are described. A fuzzy relation algebra $\mathcal{F}$, which will be defined below, is an algebraic structure over a nonempty set $\mathcal{F}$. In other words fuzzy relation algebras are fuzzy algebras with composition and involution. Elements of $\mathcal{F}$ are denoted by Greek letters such as $\alpha, \beta, \cdots$ and so on. The composite of a relation α followed by a relation β will be noted by $\alpha\beta$, unless confusion occurs.

Definition 3.2.1 A **fuzzy relation algebra** $\mathcal{F} = (\mathcal{F}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O, \nabla, \mathrm{id})$ is an algebraic structure over a nonempty set $\mathcal{F}$ satisfying the following:
F1. **[Fuzzy Algebra]** The tuple $(\mathcal{F}, \sqsubseteq, \sqcup, \sqcap, \cdot, O, \nabla)$ is a fuzzy algebra.
F2. **[Relation Algebra]** The tuple $(\mathcal{F}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, O, \nabla, \mathrm{id})$ is a relation algebra.
F3. **[Compatibility with Semi-Scalar Multiplication]** (a) $k(\alpha\beta) = (k\alpha)(k\beta)$, (b) $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla)$, (c) $(k\alpha)^\sharp = k\alpha^\sharp$. □

It is obvious that each algebra F-**Rel**$(X) = (F$-**Rel**$(X), \subseteq, \cup, \cap, \circ, ^\cup, *, 0_X, 1_X, E_X)$ is a fuzzy relation algebra. In this chapter all elements in a fuzzy relation algebra $\mathcal{F}$ are called "relations" for short.

Proposition 3.2.1 *Let α, β, β' be relations and k, k' scalars. Then it holds that*

$$(k\alpha)\beta \sqsubseteq k(\alpha\nabla) \quad \textit{and} \quad \alpha(k\beta) \sqsubseteq k(\nabla\beta) \ .$$

Proof. It follows from $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla) \sqsubseteq (k\alpha)(k\nabla) = k(\alpha\nabla)$ by axioms F3(b) and F3(a). □

The next proposition mentions basic properties of crisp relations.

Proposition 3.2.2 *Let α and β be relations and k a scalar. Then the following holds:*

(a) *If β is crisp, then $(k\alpha)\beta = k(\alpha\beta)$. (If α is crisp, then $\alpha(k\beta) = k(\alpha\beta)$).*

(b) *If α and β are crisp, then so are $\alpha^\sharp$ and $\alpha\beta$.*

(c) *The identity relation* id *is crisp.*

(d) *If $\nabla = \alpha \sqcup \beta$, $\alpha \sqcap \beta = O$ and $\nabla\alpha = \alpha$, then $\nabla\beta = \beta$.*

Proof. (a) Assume that β is crisp. Then $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla) = (k\alpha)(k\beta) = k(\alpha\beta)$ by axioms F3(b) and F3(a).
(b) Assume that α and β are crisp. Then $\alpha^\sharp \sqcap k\nabla = (\alpha \sqcap k\nabla)^\sharp = (k\alpha)^\sharp = k\alpha^\sharp$ by axioms R2, F3(c), propositions 2.3.1(e) and 2.3.1(a). And $\alpha\beta \sqcap k\nabla \sqsubseteq \alpha[\beta \sqcap \alpha^\sharp(k\nabla)] \sqsubseteq \alpha[\beta \sqcap k(\nabla\nabla)] \sqsubseteq \alpha(\beta \sqcap k\nabla) = \alpha(k\beta) = k(\alpha\beta)$ by axiom R3, and propositions 3.2.1, 2.3.1(a) and (a).
(c) It follows from $k\mathrm{id} \sqsubseteq \mathrm{id} \sqcap k\nabla = \mathrm{id} \sqcap k(\mathrm{id}\nabla) = \mathrm{id} \sqcap (k\mathrm{id})\nabla \sqsubseteq (k\mathrm{id})[(k\mathrm{id})^\sharp \mathrm{id} \sqcap \nabla] = k\mathrm{id}$ by axioms R3 and F3(a).
(d) It is obvious that $\beta \sqsubseteq \nabla\beta$. And by axiom R3 $\nabla\beta \sqcap \alpha \sqsubseteq \nabla(\beta \sqcap \nabla^\sharp\alpha) \sqsubseteq \nabla(\beta \sqcap \nabla\alpha) = \nabla(\beta \sqcap \alpha) = \nabla O = O$ since $\nabla\alpha = \alpha$ and $\alpha \sqcap \beta = O$. So we have $\nabla\beta = \nabla\beta \sqcap \nabla = \nabla\beta \sqcap (\alpha \sqcup \beta) = (\nabla\beta \sqcap \alpha) \sqcup (\nabla\beta \sqcap \beta) = \beta$ since $\nabla = \alpha \sqcup \beta$. □

From propositions 3.1.2(a), 3.1.2(b), 3.2.2(b), and 3.2.2(c) the set of all crisp relations in the fuzzy relation algebra $\mathcal{F}$ forms a (Boolean) relation algebra in the sense of [SS85].

3.3 Point Axiom

In this section we state a concept of point relations and a point axiom in fuzzy relation algebras. The concept of point relations defined here contains the so-called Tarski rule [Tar41] and so is stronger than that of [SS93]. Consequently, the Tarski rule is not necessary in proving the representation theorem of relation algebras.

Proposition 3.3.1 *Let α be a crisp relation such that $\nabla\alpha = \alpha$. Then the following four conditions are equivalent:*

(a) $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$.

(b) $\nabla = \alpha\alpha^\sharp$.

(c) $\nabla = \alpha\nabla$.

(d) $\alpha\nabla = k\nabla$ *for some* $k > 0$.

Proof. (a)$\Longrightarrow$(b) If $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$, then $\nabla = \nabla\mathrm{id} \sqsubseteq \nabla\alpha\alpha^\sharp = \alpha\alpha^\sharp$.
(b)$\Longrightarrow$(c) If $\nabla = \alpha\alpha^\sharp$, then $\nabla = \alpha\alpha^\sharp \sqsubseteq \alpha\nabla$.
(c)$\Longrightarrow$(a) If $\nabla = \alpha\nabla$, then $\mathrm{id} = \mathrm{id} \sqcap \nabla = \mathrm{id} \sqcap \alpha\nabla \sqsubseteq \alpha(\alpha^\sharp\mathrm{id} \sqcap \nabla) = \alpha\alpha^\sharp$.
(d)$\Longrightarrow$(c) If $k\nabla \sqsubseteq \alpha\nabla$ for $k > 0$, then $\nabla \sqsubseteq \alpha\nabla$ by proposition 3.1.2(c) since $\alpha\nabla$ is crisp.
(c)$\Longrightarrow$(d) It is trivial. □

In view of Lemma 2.1.1 the concept of point relations in fuzzy relation algebras is defined as follows:

Definition 3.3.1 A **point relation** x is a crisp relation such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\mathrm{id} \sqsubseteq xx^\sharp$ and $\nabla x = x$. (Point relations will be denoted by lower case Roman letters such as $x, y, z, \cdots$.) □

Note that a point relation x is nonzero from its totality $\mathrm{id} \sqsubseteq xx^\sharp$. For two point relations x, y a relation $x^\sharp y$ is called a pair relation. Every pair relation $x^\sharp y$ is nonzero since $y \sqsubseteq x(x^\sharp y)$ by the totality $\mathrm{id} \sqsubseteq xx^\sharp$ of x. The point axiom, which will be stated in definition 3.3.2, asserts that pair relations play the rôle of atomic relations. The last proposition 3.3.1 indicates that the Tarski rule [Tar41, SS93] for point relations is equivalent to the totality [KM94]. This enables us to deduce the main theorems in section 3.4 for fuzzy relation algebras without the Tarski rule.

Proposition 3.3.2 *Let x, x_0, y, y_0 be point relations and k a scalar. Then the following holds:*

(a) *If $x \sqsubseteq y$, then $x = y$.*

(b) *If $x^\sharp y \sqsubseteq x_0^\sharp y_0$, then $x = x_0$ and $y = y_0$.*

(c) *$\alpha \sqcap x^\sharp y = k(x^\sharp y)$ if and only if $x\alpha \sqcap y = ky$.*

Proof. (a) With $\mathrm{id} \sqsubseteq xx^\sharp$, $x^\sharp \sqsubseteq y^\sharp$ and $y^\sharp y \sqsubseteq \mathrm{id}$, we have $y \sqsubseteq xx^\sharp y \sqsubseteq xy^\sharp y \sqsubseteq x$.
(b) Assume $x^\sharp y \sqsubseteq x_0^\sharp y_0$. Then $y = \nabla y = \nabla x^\sharp y \sqsubseteq \nabla x_0^\sharp y_0 = y_0$ by proposition 3.3.1 and so $y = y_0$ by (a). Similarly $x = x_0$.
(c) First note that $x\alpha \sqcap y = x(\alpha \sqcap x^\sharp y)$ and $\alpha \sqcap x^\sharp y = x^\sharp(x\alpha \sqcap y)$, because

$$x\alpha \sqcap y \sqsubseteq x(\alpha \sqcap x^\sharp y) \sqsubseteq xx^\sharp(x\alpha \sqcap y) \sqsubseteq \nabla(x\alpha \sqcap y) \sqsubseteq x\alpha \sqcap y \ ,$$

and

$$\alpha \sqcap x^\sharp y \sqsubseteq x^\sharp(x\alpha \sqcap y) \sqsubseteq x^\sharp x(\alpha \sqcap x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y$$

by $x^\sharp x \sqsubseteq \mathrm{id}$. Now assume $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Then

$$x\alpha \sqcap y = x(\alpha \sqcap x^\sharp y) = x[k(x^\sharp y)] = k[x(x^\sharp y)] = k(\nabla y) = ky \ .$$

Conversely, assume $x\alpha \sqcap y = ky$. Then

$$\alpha \sqcap x^\sharp y = x^\sharp(x\alpha \sqcap y) = x^\sharp(ky) = k(x^\sharp y)$$

since $x^\sharp$ is crisp. □

By making use of the last definition of point relations in fuzzy relation algebras, we add the following axiom:

Definition 3.3.2 A fuzzy relation algebra $\mathcal{F}$ satisfies the **point axiom** iff:
F4. For each nonzero relation α there is a scalar $k > 0$ and point relations x, y such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. □

In what follows, we assume that a fixed fuzzy relation algebra $\mathcal{F}$ satisfies the point axiom F4.

Proposition 3.3.3 *Let α and β be relations, x and y point relations, and k and k' scalars. Then the following holds:*

(a) *If α is a nonzero relation with $\nabla\alpha = \alpha$, then there exist a scalar $k > 0$ and a point relation y such that $ky \sqsubseteq \alpha$.*

(b) *If $k\alpha \sqsubseteq k'\beta$ for a nonzero crisp relation α, then $k \leq k'$.*

(c) *If $\alpha \sqsubseteq x^\sharp y$, then there is a unique scalar k such that $\alpha = k(x^\sharp y)$.*

(d) *If $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, then $k = \max\{r \in [0,1] \mid r(x^\sharp y) \sqsubseteq \alpha\}$.*

(e) *If $x \neq y$, then $x \sqcap y = O$ and $xy^\sharp = O$.*

Proof. (a) From the point axiom F4 there exist a scalar $k > 0$, and point relations x, y such that $k(x^\sharp y) \sqsubseteq \alpha$. Hence $ky = k(\nabla y) = k(xx^\sharp y) = x[k(x^\sharp y)] \sqsubseteq \nabla\alpha = \alpha$ by propositions 3.3.1 and 3.2.2(a).
(b) First we will show that $k\nabla \sqsubseteq k'\nabla$. By the point axiom F4, there are a scalar $r > 0$ and point relations x, y such that $r(x^\sharp y) \sqsubseteq \alpha$. But $x^\sharp y \sqsubseteq \alpha$ by proposition 3.1.2(c) since α is crisp. Hence, $\nabla = \nabla(x^\sharp y)\nabla \sqsubseteq \nabla\alpha\nabla$ by proposition 3.3.1 and $k\nabla \sqsubseteq k(\nabla\alpha\nabla) = \nabla(k\alpha)\nabla \sqsubseteq \nabla(k'\beta)\nabla = k'(\nabla\beta\nabla) \sqsubseteq k'\nabla$. Now assume $k' < k$. Then $k'/k \in [0,1]$ and $k\nabla \sqsubseteq k[(k'/k)\nabla]$ and so $\nabla \sqsubseteq (k'/k)\nabla$ by axiom A2(e). Thus $k'/k = 1$ follows from by proposition 3.1.1(d), which is a contradiction. Therefore, $k \leq k'$.
(c) It is trivial that if $\alpha = O$ then $\alpha = 0(x^\sharp y)$ by axiom A2(a). Next assume $\alpha \neq O$. Then by the point axiom F4, there are a scalar $k > 0$ and point relations x_0, y_0 such that $\alpha \sqcap x_0^\sharp y_0 = k(x_0^\sharp y_0)$. Hence, $k(x_0^\sharp y_0) \sqsubseteq \alpha \sqsubseteq x^\sharp y$, and so $x = x_0$ and $y = y_0$ by propositions 3.1.2(c) and 3.3.2(b), which implies $\alpha = k(x^\sharp y)$. The uniqueness of k follows from (b).
(d) Define a set $M = \{r \in [0,1] \mid r(x^\sharp y) \sqsubseteq \alpha\}$ of scalars. If $r \in M$, then $r(x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y = k(x^\sharp y)$ and so $r \leq k$ by (b). On the other hand, $k \in M$ follows from $k(x^\sharp y) = \alpha \sqcap x^\sharp y \sqsubseteq \alpha$. Hence k is the maximum element of M.
(e) Assume that $x \neq y$ and $x \sqcap y \neq O$. Note that $\nabla(x \sqcap y) = x \sqcap y$ from $x \sqcap y \sqsubseteq \nabla(x \sqcap y) \sqsubseteq \nabla x \sqcap \nabla y = x \sqcap y$. Then by (a) there are a scalar $r > 0$ and a point relation z such that $rz \sqsubseteq x \sqcap y$. But $x \sqcap y$ is crisp by proposition 3.1.2(a). Hence we have $z \sqsubseteq x \sqcap y$ by axiom A1 and proposition 3.1.2(c) and so $x = z = y$ by proposition 3.3.2(a). Finally if $x \sqcap y = O$, then $xy^\sharp = xy^\sharp \sqcap \nabla \sqsubseteq (x \sqcap \nabla y)y^\sharp = (x \sqcap y)y^\sharp = O$. □

Throughout the rest of this chapter, the set of all point relations in $\mathcal{F}$ will be denoted by X.

Proposition 3.3.4 *For the set X of all point relations in $\mathcal{F}$, the following holds:*

(a) $\nabla = \sqcup_{z \in X} z$.

(b) $\mathrm{id} = \sqcup_{z \in X} z^\sharp z$.

Proof. (a) Set $\alpha = \sqcup_{z \in X} z$. It is clear that $\alpha \sqsubseteq \nabla$. As point relations are crisp by the definition, α is also crisp by proposition 3.1.2(a) and by axiom A2 there is a relation β such that $\nabla = \alpha \sqcup \beta$ and $\alpha \sqcap \beta = O$. Then $\nabla\beta = \beta$ by proposition 3.2.2(d). Assume $\beta \neq O$. By proposition 3.3.3(a) there are a scalar $k > 0$ and a point relation y such that $ky \sqsubseteq \beta$. But β is crisp by proposition 3.1.2(b) and so $y \sqsubseteq \beta$ by proposition 3.1.2(c). Therefore, $y \sqsubseteq \alpha \sqcap \beta = O$, which contradicts to $y \neq O$. This proves $\beta = O$.
(b) It is clear that $\sqcup_{z \in X} z^\sharp z \sqsubseteq \mathrm{id}$. By using (a), also we have

$$\mathrm{id} = \mathrm{id} \sqcap \nabla = \sqcup_{z \in X} (\mathrm{id} \sqcap z) \sqsubseteq \sqcup_{z \in X} (\mathrm{id} z^\sharp \sqcap \mathrm{id}) z \sqsubseteq \sqcup_{z \in X} z^\sharp z \ . \qquad \square$$

From the point axiom F4 there exists a scalar k such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. By proposition 3.3.3(c) such scalar k is unique. For a relation α and point relations x, y a scalar $\chi(\alpha)(x, y)$ is defined to be the unique scalar k with $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Thus by proposition 3.3.2(c), $\chi(\alpha)(x, y)$ is the unique scalar such that

$$x\alpha \sqcap y = \chi(\alpha)(x, y) y \ .$$

Hence, $\chi(\alpha)$ defines a fuzzy relation on the set of all point relations in $\mathcal{F}$. In the next section, we will prove that the function $\chi : \mathcal{F} \to F\text{-}\mathbf{Rel}(X)$ is an isomorphism of fuzzy relation algebras.

Proposition 3.3.5 *If $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, then $r\alpha \sqcap k(x^\sharp y) = rk(x^\sharp y)$ for all scalars r.*

Proof. If $r = 0$, then the assertion is trivial. So we assume $r > 0$. As $r\alpha \sqcap k(x^\sharp y) \sqsubseteq k(x^\sharp y) \sqsubseteq x^\sharp y$, by proposition 3.3.3(c) there is a scalar r' such that $r\alpha \sqcap k(x^\sharp y) = r'(x^\sharp y)$. From $k(x^\sharp y) \sqsubseteq \alpha$ it follows that $rk(x^\sharp y) \sqsubseteq r\alpha \sqcap k(x^\sharp y) = r'(x^\sharp y)$ and so $rk \leq r'$ by proposition 3.3.3(b). Also $r' \leq r$ follows from $r'(x^\sharp y) \sqsubseteq r\alpha$. Since $0 \leq r'/r \leq 1$ we have $r(r'/r)(x^\sharp y) = r'(x^\sharp y) \sqsubseteq r\alpha$ and $(r'/r)(x^\sharp y) \sqsubseteq \alpha$ by axiom A2(e). Hence $(r'/r)(x^\sharp y) \sqsubseteq k(x^\sharp y)$ and $r'/r \leq k$. This proves $r' = rk$. $\square$

3.4 Representation Theorem

First we prove the representation theorem for fuzzy relation algebras asserting that every relation in a fuzzy relation algebra satisfying the point axiom F4, can be represented as a union of pairs of point relations with semi-scalar weights.

Theorem 3.4.1 (Representation Theorem) *Let X be the set of all point relations in $\mathcal{F}$. Then every relation α has a unique representation*

$$\alpha = \sqcup_{x,y \in X} \chi(\alpha)(x, y)(x^\sharp y) \ .$$

Proof. Since $\nabla = \sqcup_{y \in X} y$ by proposition 3.3.4(a), we have

$$\begin{aligned} x\alpha &= x\alpha \sqcap \nabla \\ &= x\alpha \sqcap (\sqcup_{y \in X} y) \\ &= \sqcup_{y \in X} (x\alpha \sqcap y) \\ &= \sqcup_{y \in X} \chi(\alpha)(x, y) y \end{aligned}$$

and so

$$\begin{aligned} \alpha &= \mathrm{id}\alpha \\ &= \sqcup_{x \in X} x^\sharp x\alpha \\ &= \sqcup_{x \in X} x^\sharp [\sqcup_{y \in X} \chi(\alpha)(x, y) y] \\ &= \sqcup_{x \in X} \sqcup_{y \in X} x^\sharp [\chi(\alpha)(x, y) y] \\ &= \sqcup_{x,y \in X} \chi(\alpha)(x, y)(x^\sharp y) \end{aligned}$$

using $\mathrm{id} = \sqcup_{x \in X} x^\sharp x$ by proposition 3.3.4(b). Finally, we show the uniqueness of the representation. Assume $\alpha = \sqcup_{x,y \in X} k_{x,y}(x^\sharp y)$. Then for all $x_0, y_0 \in X$, we have

$$x_0\alpha = \sqcup_{x,y \in X} k_{x,y}(x_0 x^\sharp y) = \sqcup_{y \in X} k_{x_0,y} y$$

and

$$x_0\alpha \sqcap y_0 = \sqcup_{y\in X}[k_{x_0,y}y \sqcap y_0] = k_{x_0,y_0}y_0 \ .$$

Hence, $k_{x,y} = \chi(\alpha)(x,y)$ by proposition 3.3.3(b). □

Corollary 3.4.1 *For every fuzzy relation algebra $\mathcal{F}$ the function $\chi : \mathcal{F} \to F\text{-}\mathbf{Rel}(X)$ is a bijection.*

Proof. If $\chi(\alpha) = \chi(\beta)$, then by the last theorem we have

$$\alpha = \sqcup_{x,y\in X}\chi(\alpha)(x,y)(x^\sharp y) = \sqcup_{x,y\in X}\chi(\beta)(x,y)(x^\sharp y) = \beta \ ,$$

which shows that χ is injective. Given a fuzzy relation $R \in F\text{-}\mathbf{Rel}(X)$ we set $\alpha_R = \sqcup_{x,y\in X}R(x,y)(x^\sharp y)$. Then by the uniqueness of representation in the last theorem we have $R(x,y) = \chi(\alpha_R)(x,y)$, which means that χ is surjective. □

Theorem 3.4.2 *For a relation α and a set $\{k_\lambda\}_\lambda$ of scalars the identity $\sqcup_\lambda k_\lambda\alpha = (\vee_\lambda k_\lambda)\alpha$ holds.*

Proof. First we show that $\sqcup_\lambda k_\lambda(x^\sharp y) = (\vee_\lambda k_\lambda)(x^\sharp y)$ for point relations x, y. It is obvious that $\sqcup_\lambda k_\lambda(x^\sharp y) \sqsubseteq (\vee_\lambda k_\lambda)(x^\sharp y)$ and so there is a scalar k such that $\sqcup_\lambda k_\lambda(x^\sharp y) = k(x^\sharp y)$ by proposition 3.3.3(c). Then $k \leq \vee_\lambda k_\lambda$ by $k(x^\sharp y) \sqsubseteq (\vee_\lambda k_\lambda)(x^\sharp y)$ and proposition 3.3.3(b). On the other hand, $k_\lambda \leq k$ from $k_\lambda(x^\sharp y) \sqsubseteq k(x^\sharp y)$, and so $\vee_\lambda k_\lambda \leq k$. Therefore this proves $k = \vee_\lambda k_\lambda$. We are now ready to prove the general case. Since $\alpha = \sqcup_{x,y\in X}\chi(\alpha)(x,y)(x^\sharp y)$, by the representation theorem 3.4.1 we have

$$\begin{aligned}
\sqcup_\lambda k_\lambda\alpha &= \sqcup_\lambda k_\lambda[\sqcup_{x,y\in X}\chi(\alpha)(x,y)(x^\sharp y)] \\
&= \sqcup_\lambda \sqcup_{x,y\in X}[k_\lambda\chi(\alpha)(x,y)(x^\sharp y)] \\
&= \sqcup_{x,y\in X}\sqcup_\lambda[k_\lambda\chi(\alpha)(x,y)(x^\sharp y)] \\
&= \sqcup_{x,y\in X}[(\vee_\lambda k_\lambda)\chi(\alpha)(x,y)(x^\sharp y)] \\
&= (\vee_\lambda k_\lambda)[\sqcup_{x,y\in X}\chi(\alpha)(x,y)(x^\sharp y)] \\
&= (\vee_\lambda k_\lambda)\alpha \ .
\end{aligned}$$

□

The following theorem corresponds to the Tarski rule for Boolean relation algebras [SS85, Tar41].

Theorem 3.4.3 *For a nonzero relation α in $\mathcal{F}$ there is a scalar $k > 0$ such that $\nabla\alpha\nabla = k\nabla$.*

Proof. By means of the representation theorem 3.4.1 we have

$$\begin{aligned}
\nabla\alpha\nabla &= \sqcup_{x,y\in X}\nabla[\chi(\alpha)(x,y)(x^\sharp y)]\nabla \\
&= \sqcup_{x,y\in X}\chi(\alpha)(x,y)[\nabla(x^\sharp y)\nabla] \\
&= \sqcup_{x,y\in X}\chi(\alpha)(x,y)\nabla\nabla \\
&= \sqcup_{x,y\in X}\chi(\alpha)(x,y)\nabla \\
&= (\vee_{x,y\in X}\chi(\alpha)(x,y))\nabla \ .
\end{aligned}$$

Set $k = \vee_{x,y\in X}\chi(\alpha)(x,y)$. Then $\nabla\alpha\nabla = k\nabla$. By the way there exist point relations x, y such that $\chi(\alpha)(x,y) > 0$ by the point axiom F4. Hence $k > 0$, which completes the proof. □

The following proposition shows that $\chi : \mathcal{F} \to F\text{-}\mathbf{Rel}(X)$ preserves all operations of fuzzy relations, that is, χ is a homomorphism of fuzzy relation algebras.

Proposition 3.4.1 *Let α, β be relations and k a scalar. Then the following holds:*

(a) $\chi(O) = 0_X$, $\chi(\nabla) = 1_X$ *and* $\chi(\mathrm{id}) = E_X$.

(b) *If $\alpha \sqsubseteq \beta$, then $\chi(\alpha) \subseteq \chi(\beta)$.*

(c) $\chi(\alpha \sqcup \beta) = \chi(\alpha) \cup \chi(\beta)$.

(d) $\chi(\alpha \sqcap \beta) = \chi(\alpha) \cap \chi(\beta)$.

(e) $\chi(\alpha^\sharp) = \chi(\alpha)^\cup$.

(f) $\chi(k\alpha) = k\chi(\alpha)$.

(g) $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$.

Proof. (a) The fist follows from $xO \sqcap y = O = 0_X y$, the second follows from $x\nabla \sqcap y = \nabla \sqcap y = y$ by $x\nabla = \nabla$, and the last follows from $x\mathrm{id} \sqcap y = x \sqcap y$ and proposition 3.3.3(e).
(b) $\chi(\alpha)(x,y)y = x\alpha \sqcap y \sqsubseteq x\beta \sqcap y = \chi(\beta)(x,y)y$ and so $\chi(\alpha)(x,y) \leq \chi(\beta)(x,y)$ by proposition 3.3.3(b).
(c) follows from

$$\begin{aligned}
\chi(\alpha \sqcup \beta)(x,y)y &= x(\alpha \sqcup \beta) \sqcap y \\
&= (x\alpha \sqcup x\beta) \sqcap y \\
&= (x\alpha \sqcap y) \sqcup (x\beta \sqcap y) \\
&= \chi(\alpha)(x,y)y \sqcup \chi(\beta)(x,y)y \\
&= [\chi(\alpha)(x,y) \vee \chi(\beta)(x,y)]y \\
&= [\chi(\alpha) \cup \chi(\beta)](x,y)y \ .
\end{aligned}$$

(d) First note that $x(\alpha \sqcap \beta) = x\alpha \sqcap x\beta$. It follows from $x^\sharp x \sqsubseteq \mathrm{id}$ that

$$x(\alpha \sqcap \beta) \sqsubseteq x\alpha \sqcap x\beta \sqsubseteq x(\alpha \sqcap x^\sharp x\beta) \sqsubseteq x(\alpha \sqcap \beta) \ .$$

Hence, we have

$$\begin{aligned}
\chi(\alpha \sqcap \beta)(x,y)y &= x(\alpha \sqcap \beta) \sqcap y \\
&= (x\alpha \sqcap x\beta) \sqcap y \\
&= (x\alpha \sqcap y) \sqcap (x\beta \sqcap y) \\
&= \chi(\alpha)(x,y)y \sqcap \chi(\beta)(x,y)y \\
&= [\chi(\alpha)(x,y) \wedge \chi(\beta)(x,y)]y \\
&= [\chi(\alpha) \cap \chi(\beta)](x,y)y \ .
\end{aligned}$$

(e) The identity $\chi(\alpha^\sharp) = \chi(\alpha)^\cup$ follows from

$$\begin{aligned}
x\alpha \sqcap y = ky &\iff \alpha \sqcap x^\sharp y = k(x^\sharp y) \\
&\iff \alpha^\sharp \sqcap y^\sharp x = k(y^\sharp x) \\
&\iff y\alpha^\sharp \sqcap x = kx \ .
\end{aligned}$$

(f) It follows from

$$\begin{aligned}
\chi(k\alpha)(x,y)y &= x(k\alpha) \sqcap y \\
&= k(x\alpha) \sqcap k\nabla \sqcap y \quad (\text{ by } \ k(x\alpha) \sqsubseteq k\nabla \ ) \\
&= k(x\alpha) \sqcap ky \\
&= k(x\alpha \sqcap y) \\
&= k[\chi(\alpha)(x,y)]y \\
&= [k\chi(\alpha)(x,y)]y \ .
\end{aligned}$$

(g) First we have $\chi(\alpha\beta)(x,y) = \vee_z \chi(\alpha z^\sharp z\beta)(x,y)$ from

$$\begin{aligned}
\chi(\alpha\beta)(x,y)y &= x\alpha\beta \sqcap y \\
&= x\alpha\mathrm{id}\beta \sqcap y \\
&= x\alpha(\sqcup_z z^\sharp z)\beta \sqcap y \\
&= \sqcup_z (x\alpha z^\sharp z\beta \sqcap y) \\
&= \sqcup_z \chi(\alpha z^\sharp z\beta)(x,y)y \\
&= [\vee_z \chi(\alpha z^\sharp z\beta)(x,y)]y \quad (\text{ by Theorem 3.4.2 }) \ .
\end{aligned}$$

To complete the proof it suffices to show

$$\chi(\alpha z^\sharp z\beta)(x,y) = \chi(\alpha)(x,z) \wedge \chi(\beta)(z,y) \ .$$

Note that $x\alpha \sqcap z = x\alpha z^\sharp z$ (by $x\alpha z^\sharp z \sqsubseteq x\alpha \sqcap \nabla z \sqsubseteq (x\alpha z^\sharp \sqcap \nabla)z = x\alpha z^\sharp z$) and so

$$\begin{aligned}
[\chi(\alpha)(x,z) \wedge \chi(\beta)(z,y)]z &= \chi(\alpha)(x,z)z \sqcap \chi(\beta)(z,y)z \quad (\text{ by A2(c) }) \\
&= \chi(\alpha)(x,z)z \sqcap \chi(\beta^\sharp)(y,z)z \\
&= x\alpha \sqcap z \sqcap y\beta^\sharp \\
&= x\alpha z^\sharp z \sqcap y\beta^\sharp \ .
\end{aligned}$$

Set $r = \chi(\alpha z^\sharp z\beta)(x,y)$ and $s = \chi(\alpha)(x,z) \wedge \chi(\beta)(z,y)$. Then $x\alpha z^\sharp z\beta \sqcap y = ry$ and $x\alpha z^\sharp z \sqcap y\beta^\sharp = sz$. We have $r \leq s$ from

$$ry = x\alpha z^\sharp z\beta \sqcap y \sqsubseteq (x\alpha z^\sharp z \sqcap y\beta^\sharp)\beta \sqsubseteq (sz)\beta \sqsubseteq s(z\nabla)$$

and $s \leq r$ from

$$sz = x\alpha z^\sharp z \sqcap y\beta^\sharp \sqsubseteq (x\alpha z^\sharp z\beta \sqcap y)\beta^\sharp \sqsubseteq (ry)\beta^\sharp \sqsubseteq r(y\nabla) \ .$$

Therefore, $r = s$. □

The following is immediately deduced from the last proposition.

Corollary 3.4.2 (Insertion Theorem) *Let α and β be relations and let x and y be point relations. If $\chi(\alpha\beta)(x,y) > 0$, then there is a point relation z such that $\chi(\alpha)(x,z) > 0$ and $\chi(\beta)(z,y) > 0$.* □

It is now obvious that χ^{-1} is a function and is a homomorphism from the algebra F-**Rel**(X) of all fuzzy relations to $\mathcal{F}$. Consequently the following corollary is deduced:

Corollary 3.4.3 (Isomorphism Theorem) *Every fuzzy relation algebra $\mathcal{F}$ satisfying the point axiom is isomorphic to the algebra F-**Rel**(X) of all fuzzy relations on the set X of all point relations of $\mathcal{F}$.* □

The reader can easily understand that all results of the paper remain valid, even if the unit interval $[0,1]$ is replaced by a subset $\mathcal{L}$ of $[0,1]$ such that (0) $0, 1 \in \mathcal{L}$, (i) If $\{k_\lambda\}_\lambda \subseteq \mathcal{L}$, then $\vee_\lambda k_\lambda \in \mathcal{L}$ and $\wedge_\lambda k_\lambda \in \mathcal{L}$, (ii) If $k, k' \in \mathcal{L}$, then $kk' \in \mathcal{L}$, and (iii) If $k, k' \in \mathcal{L}$ and $k < k'$, then $k = k'k''$ for some $k'' \in \mathcal{L}$. For example, $\mathcal{L} = \{0\} \cup \{k^n \mid n = 0, 1, 2, \cdots\}$ $(0 < k \leq 1)$.

Chapter 4

An Algebraic Study on Cartesian Products of Fuzzy Relations

Jónsson and Tarski [JT52] characterized cartesian products of Boolean relation algebras by using a notion of ideal relations. Ideal relations are universal with respect to the composition with the universal relation ∇. A representable Boolean relation algebra which is isomorphic to the algebra of all ordinary homogeneous relations has no ideal relation except for the zero relation O and the universal relation ∇.

The aim of this chapter is to provide an algebraic characterisation of cartesian products of fuzzy relations by adding two axioms to a set of axioms of fuzzy relation algebras given in section 3.2. The former is called an axiom of cartesian products, which corresponds to the algebraic characterisation of cartesian products of Boolean relation algebras introduced by Jónsson and Tarski [JT52], and the other is called a point axiom for cartesian products. Ideal relations in fuzzy relation algebras are required to be crisp in order to keep their general properties the same as the properties of such relations in Boolean relation algebras. The point axiom for cartesian products is defined by using a new notion of point relations in a fuzzy relation algebra $\mathcal{F}$ satisfying the axiom of cartesian products, as an extension of point axiom provided in section 3.3, since such point relations and the point axiom don't work on cartesian products of fuzzy relation algebras.

Section 4.1 provides a definition and some properties of ideal relations in fuzzy relation algebras. In section 4.2, we define the axiom P1 of cartesian products. Axiom P1 provide a characterisation of cartesian products of fuzzy relation algebras. In section 4.3, we first define a new concept of point relations for cartesian products of fuzzy relation algebras. Using this concept, the point axiom P2 for cartesian products is defined. Finally a representation theorem for fuzzy relation algebras satisfying the axiom P1 of cartesian products and the point axiom P2 for cartesian products is proved.

This chapter is based on [Fur97a].

4.1 Ideal Relations

Jónsson and Tarski [JT52] investigated ideal relations in Boolean relation algebras. In this section we define ideal relations in fuzzy relation algebras, and consider some properties of ideal relations. Though in Boolean relation algebras ideal relations are just universal with respect to the composition with the universal relation ∇, ideal relations are also required to be crisp in fuzzy relation algebras. In the next section, ideal relations play an important rôle.

Definition 4.1.1 A fuzzy relation ξ is an **ideal** if ξ is crisp and $\nabla\xi\nabla = \xi$. □

It is obvious that $\nabla\nabla\nabla = \nabla$. So the universal relation ∇ is ideal. And so the zero relation O is. From theorem 3.4.3, fuzzy relation algebras satisfying the point axiom F4 have no ideal relation except for the zero relation O and the universal relation ∇.

Proposition 4.1.1 *Let ξ, η, be ideal, α, β relations and k a scalar. Then the following holds:*

(a) $\xi \sqcup \eta$, $\xi \sqcap \eta$ *and* $\xi^\sharp$ *are ideal.*

(b) $\xi = \xi\nabla = \nabla\xi$.

(c) $\xi = \xi^\sharp$.

(d) $\xi\eta = \xi \sqcap \eta$.

(e) $\alpha\beta \sqcap \xi = (\alpha \sqcap \xi)(\beta \sqcap \xi)$.

Proof. (a) The first follows from $\nabla(\xi \sqcup \eta)\nabla = \nabla\xi\nabla \sqcup \nabla\eta\nabla = \xi \sqcup \eta$, the second follows from $\nabla(\xi \sqcap \eta)\nabla \sqsubseteq \nabla\xi\nabla \sqcap \nabla\eta\nabla = \xi \sqcap \eta = \xi \sqcap \nabla\eta\nabla \sqsubseteq \nabla(\nabla^\sharp\xi\nabla^\sharp \sqcap \eta)\nabla = \nabla(\xi \sqcap \eta)\nabla$, and the last follows from $\nabla\xi^\sharp\nabla = (\nabla\xi\nabla)^\sharp = \xi^\sharp$ by axioms R3 and R2.
(b) follows from $\nabla\xi = \nabla\nabla\xi\nabla = \xi$.
(c) $\xi^\sharp = \xi^\sharp\nabla = \xi^\sharp\nabla \sqcap \nabla \sqsubseteq \xi^\sharp(\nabla \sqcap \xi\nabla) \sqsubseteq \nabla\xi\nabla = \xi$ by (b). Similarly it is shown that $\xi \sqsubseteq \xi^\sharp$ holds.
(d) $\xi\eta = \xi\eta \sqcap \nabla \sqsubseteq (\xi \sqcap \nabla\eta^\sharp)\eta \sqsubseteq (\xi \sqcap \eta)\nabla = \xi \sqcap \eta$ by axiom R3, (b), (c), and (a). Conversely $\xi \sqcap \eta = \xi \sqcap \nabla\eta \sqsubseteq (\xi\eta^\sharp \sqcap \nabla)\eta \sqsubseteq \xi\eta$ by (b), axiom R3, and (c).
(e) It follows from

$$\begin{array}{rcl} \alpha\beta \sqcap \xi & \sqsubseteq & (\alpha \sqcap \xi\beta^\sharp)\beta \\ & = & (\alpha \sqcap \xi\beta^\sharp)\beta \sqcap \xi \\ & \sqsubseteq & (\alpha \sqcap \xi\beta^\sharp)[\beta \sqcap (\alpha \sqcap \xi\beta^\sharp)^\sharp\xi] \\ & \sqsubseteq & (\alpha \sqcap \xi\beta^\sharp)(\beta \sqcap \alpha^\sharp\xi) \\ & \sqsubseteq & (\alpha \sqcap \xi\nabla)(\beta \sqcap \nabla\xi) \\ & = & (\alpha \sqcap \xi)(\beta \sqcap \xi) \\ & \sqsubseteq & \alpha\beta \sqcap \xi\xi \\ & \sqsubseteq & \alpha\beta \sqcap \xi \end{array}$$

by axiom R3 and (b). □

Now we consider the function $\phi_\xi : \mathcal{F} \to \mathcal{F}$ such that $\phi_\xi(\alpha) = \alpha \sqcap \xi$ for a fixed ideal relation ξ and any relation α. The following proposition shows that $\phi_\xi : \mathcal{F} \to \mathcal{F}$ preserves all operations of fuzzy relation algebras except for nullary operations.

Proposition 4.1.2 *Let ξ be an ideal relation, α, β relations and k a scalar. Then the following holds:*

(a) *If $\alpha \sqsubseteq \beta$, then $\phi_\xi(\alpha) \sqsubseteq \phi_\xi(\beta)$.*

(b) $\phi_\xi(\alpha \sqcup \beta) = \phi_\xi(\alpha) \sqcup \phi_\xi(\beta)$ *and* $\phi_\xi(\alpha \sqcap \beta) = \phi_\xi(\alpha) \sqcap \phi_\xi(\beta)$.

(c) $\phi_\xi(\alpha^\sharp) = \phi_\xi(\alpha)^\sharp$.

(d) $\phi_\xi(k\alpha) = k\phi_\xi(\alpha)$.

(e) $\phi_\xi(\alpha\beta) = \phi_\xi(\alpha)\phi_\xi(\beta)$.

Proof. (a) Assume that $\alpha \sqsubseteq \beta$, then $\phi_\xi(\alpha) = \alpha \sqcap \xi \sqsubseteq \beta \sqcap \xi = \phi_\xi(\beta)$.
(b) It follows from $\phi_\xi(\alpha \sqcup \beta) = (\alpha \sqcup \beta) \sqcap \xi = (\alpha \sqcap \xi) \sqcup (\beta \sqcap \xi) = \phi_\xi(\alpha) \sqcup \phi_\xi(\beta)$. Similarly it is shown that $\phi_\xi(\alpha \sqcap \beta) = \phi_\xi(\alpha) \sqcap \phi_\xi(\beta)$ holds.
(c) $\phi_\xi(\alpha^\sharp) = \alpha^\sharp \sqcap \xi = (\alpha \sqcap \xi)^\sharp = \phi_\xi(\alpha)^\sharp$ by proposition 4.1.1(c).
(d) Since ξ is crisp, we have $\phi_\xi(k\alpha) = k\alpha \sqcap \xi = k(\alpha \sqcap \nabla) \sqcap \xi = k\alpha \sqcap (k\nabla \sqcap \xi) = k\alpha \sqcap k\xi = k(\alpha \sqcap \xi) = k\phi_\xi(\alpha)$.
(e) $\phi_\xi(\alpha\beta) = \alpha\beta \sqcap \xi = (\alpha \sqcap \xi)(\beta \sqcap \xi) = \phi_\xi(\alpha)\phi_\xi(\beta)$ by proposition 4.1.1(e). □

If ξ is not crisp, proposition 4.1.2(d) doesn't hold. Because there is a scalar k such that $k\nabla \sqcap \xi \neq k\xi$ by the definition of crisp relations. That is a difference between our concept of ideal relations and the concept in [JT52].

For a fixed relation ξ, we let $\mathcal{F}(\xi)$ denote the set of all relations α such that $\alpha \sqsubseteq \xi$. If ξ is an ideal relation, $\mathcal{F}(\xi)$ is the image of the function ϕ_ξ which appeared before proposition 4.1.2. Note that $\mathrm{id} \sqcap \xi \in \mathcal{F}(\xi)$ holds since it holds that

$$\alpha(\mathrm{id} \sqcap \xi) \sqsubseteq \alpha\mathrm{id} = \alpha$$

and that

$$\alpha = \alpha\mathrm{id} = \alpha\mathrm{id} \sqcap \nabla \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp\xi) \sqsubseteq \alpha(\mathrm{id} \sqcap \xi\xi) \sqsubseteq \alpha(\mathrm{id} \sqcap \xi\nabla) = \alpha(\mathrm{id} \sqcap \xi)$$

for each $\alpha \in \mathcal{F}(\xi)$. Then the tuple $(\mathcal{F}(\xi), \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O, \mathrm{id} \sqcap \xi, \xi)$ is a fuzzy relation algebra. Consequently the function ϕ_ξ is a homomorphism from $\mathcal{F}$ onto $\mathcal{F}(\xi)$.

4.2 Cartesian Products of Fuzzy Relation Algebras

Jónsson and Tarski [JT52] showed that a cartesian product of Boolean relation algebras is again a Boolean relation algebra. And they also characterized cartesian products of Boolean relation algebras. It is clear that a cartesian product of fuzzy relation algebras is again a fuzzy relation algebra if we consider a relation $\alpha = \prod_{i\in I} \alpha_i$ and semi-scalar multiplication $k\alpha = k \prod_{i\in I} \alpha_i = \prod_{i\in I} k\alpha_i$, where the symbol $\prod$ is used as a cartesian product notation, and I is a finite set. (Throughout the rest of this chapter we use the symbol $\prod$ as a cartesian product notation.) In this section we characterize cartesian products of fuzzy relation algebras.

Definition 4.2.1 Let I be an arbitrary finite set. Then a fuzzy relation algebra $\mathcal{F}$ satisfies the **axiom of cartesian products** iff:
P1. There exists a finite number of ideal relations $\nabla_i \neq O$ (for each $i \in I$) such that:
(a) $\sqcup_{i\in I}\nabla_i = \nabla$. (b) If $i, j \in I$ and $i \neq j$, then $\nabla_i \sqcap \nabla_j = O$. □

Example 4.2.1 Consider an algebra $\mathcal{F}^2$ of a cartesian product $\mathcal{F} \times \mathcal{F}$ of a fuzzy relation algebra $\mathcal{F}$. It is clear that $\mathcal{F}^2$ is a fuzzy relation algebra with the zero relation $\langle O, O\rangle$ and the universal relation $\langle \nabla, \nabla\rangle$, written by O^2 and ∇^2, respectively. Let $\nabla_1 = \langle \nabla, O\rangle$ and $\nabla_2 = \langle O, \nabla\rangle$. Then both of ∇_1 and ∇_2 are ideal relations with $\nabla_1 \neq O^2$ and $\nabla_2 \neq O^2$ since $\nabla^2\nabla_1\nabla^2 = \langle \nabla\nabla\nabla, \nabla O\nabla\rangle = \langle \nabla, O\rangle = \nabla_1$ and $\nabla^2\nabla_2\nabla^2 = \langle \nabla O\nabla, \nabla\nabla\nabla\rangle = \langle O, \nabla\rangle = \nabla_2$. Also the condition (a) of axiom P1 follows from $\nabla_1 \sqcup \nabla_2 = \langle \nabla \sqcup O, O \sqcup \nabla\rangle = \langle \nabla, \nabla\rangle = \nabla^2$, and the condition (b) of axiom P1 follows from $\nabla_1 \sqcap \nabla_2 = \langle \nabla \sqcap O, O \sqcap \nabla\rangle = \langle O, O\rangle = O^2$. Therefore a fuzzy relation algebra $\mathcal{F}^2$ satisfies axiom P1 of cartesian products. □

Let $\mathcal{F}$ be a fuzzy relation algebra satisfying axiom P1 of cartesian products and ϕ_i a function from $\mathcal{F}$ to $\mathcal{F}(\nabla_i)$ such that $\phi_i(\alpha) = \alpha \sqcap \nabla_i$ for each $i \in I$ and each relation $\alpha \in \mathcal{F}$. Set $O_i = O \sqcap \nabla_i$ and $\mathrm{id}_i = \mathrm{id} \sqcap \nabla_i$ for each $i \in I$. Then the tuple $(\mathcal{F}(\nabla_i), \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O_i, \mathrm{id}_i, \nabla_i)$ is a fuzzy relation algebra, since ∇_i is an ideal relation. So the function ϕ_i is homomorphism from $\mathcal{F}$ onto $\mathcal{F}(\nabla_i)$.

Proposition 4.2.1 *Let α, β be relations in a fuzzy relation algebra $\mathcal{F}$ satisfying the axiom of cartesian products, and $i, j \in I$. If $\alpha \sqsubseteq \nabla_i$ and $\beta \sqsubseteq \nabla_j$ with $i \neq j$, then $\alpha\beta = O$.*

Proof. If $i, j \in I$ and if $i \neq j$, then $\nabla_i\nabla_j = \nabla_i \sqcap \nabla_j = O$ by proposition 4.1.1(d) and axiom P1(b). Thus $\alpha\beta = O$ since $\alpha \sqsubseteq \nabla_i$ and $\beta \sqsubseteq \nabla_j$. □

We show that every relation in a fuzzy relation algebra $\mathcal{F}$ satisfying axiom P1 of cartesian products can be represented as a supremum of relations in $\mathcal{F}(\nabla_i)$ with respect to $i \in I$.

Theorem 4.2.1 *Let $\mathcal{F}$ be a fuzzy relation algebra satisfying the axiom of cartesian products. Then every relation $\alpha \in \mathcal{F}$ has a unique representation $\alpha = \sqcup_{i\in I}\phi_i(\alpha)$.*

Proof. Since $\nabla = \sqcup_{i\in I}\nabla_i$ by axiom P1(a), we have $\alpha = \alpha \sqcap \nabla = \alpha \sqcap (\sqcup_{i\in I}\nabla_i) = \sqcup_{i\in I}(\alpha \sqcap \nabla_i) = \sqcup_{i\in I}\phi_i(\alpha)$. Next we show the uniqueness of the representation. Assume that $\alpha = \sqcup_{i\in I}\alpha_i$, where $\alpha_i \in \mathcal{F}(\nabla_i)$ for each $i \in I$.

Then $\alpha_j = \alpha_j \mathrm{id}_j = (\sqcup_{i\in I}\alpha_i)\mathrm{id}_j = [\sqcup_{i\in I}(\alpha \sqcap \nabla_i)]\mathrm{id}_j = \sqcup_{i\in I}[(\alpha \sqcap \nabla_i)\mathrm{id}_j] = \alpha \sqcap \nabla_j = \phi_j(\alpha)$ for each $j \in I$ by proposition 4.2.1. □

Now we consider a function $\psi : \mathcal{F} \to \prod_{i\in I} \mathcal{F}(\nabla_i)$ such that $\psi(\alpha) = \prod_{i\in I} \phi_i(\alpha)$.

Proposition 4.2.2 *Let $\mathcal{F}$ be a fuzzy relation algebra satisfying the axiom of cartesian products. Then the function $\psi : \mathcal{F} \to \prod_{i\in I} \mathcal{F}(\nabla_i)$ is a bijection.*

Proof. First note that $\alpha = \sqcup_{i\in I}\phi_i(\alpha)$ and $\beta = \sqcup_{i\in I}\phi_i(\beta)$ by theorem 4.2.1. If $\psi(\alpha) \sqsubseteq \psi(\beta)$, then $\alpha \sqsubseteq \beta$ since $\phi_i(\alpha) \sqsubseteq \phi_i(\beta)$ for each $i \in I$ by proposition 4.1.2(a). Thus the function ψ is injective. For a cartesian product $\prod_{i\in I} \alpha_i$ of relations $\alpha_i \in \mathcal{F}(\nabla_i)$, if we set $\alpha = \sqcup_{i\in I}\alpha_i$, then $\alpha_i = \phi_i(\alpha)$ by theorem 4.2.1. Therefore $\psi(\alpha) = \prod_{i\in I} \alpha_i$, which shows that the function ψ is surjective. □

The following proposition show that the function $\psi : \mathcal{F} \to \prod_{i\in I} \mathcal{F}(\nabla_i)$ preserves all operations of fuzzy relation algebras, that is, ψ is a homomorphism from $\mathcal{F}$ onto $\prod_{i\in I} \mathcal{F}(\nabla_i)$.

Corollary 4.2.1 *Let α, β be relations in a fuzzy relation algebra $\mathcal{F}$ satisfying the axiom of cartesian products and k a scalar. Then the following holds:*

(a) $\psi(O) = \prod_{i\in I} O_i$, $\psi(\nabla) = \prod_{i\in I} \nabla_i$ *and* $\psi(\mathrm{id}) = \prod_{i\in I} \mathrm{id}_i$.

(b) *If* $\alpha \sqsubseteq \beta$, *then* $\psi(\alpha) \sqsubseteq \psi(\beta)$

(c) $\psi(\alpha \sqcup \beta) = \psi(\alpha) \sqcup \psi(\beta)$ *and* $\psi(\alpha \sqcap \beta) = \psi(\alpha) \sqcap \psi(\beta)$.

(d) $\psi(\alpha^\sharp) = \psi(\alpha)^\sharp$.

(e) $\psi(k\alpha) = k\psi(\alpha)$.

(f) $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$.

Proof. It is trivial by the definition of ψ and proposition 4.1.2. □

Consequently we have proved that a fuzzy relation algebra $\mathcal{F}$ satisfying axiom P1 of cartesian products is isomorphic to a cartesian product $\prod_{i\in I} \mathcal{F}(\nabla_i)$ of fuzzy relation algebras $\mathcal{F}(\nabla_i)$.

4.3 Representation Theorem

In section 3.3, point relations in a fuzzy relation algebra were defined as crisp relations x such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\mathrm{id} \sqsubseteq xx^\sharp$ and $\nabla x = x$. But this definition is not suitable in the case of cartesian products of fuzzy relation algebras. In this section we provide a new definition of point relations in a fuzzy relation algebra $\mathcal{F}$ satisfying axiom P1 of cartesian products. And we call them just point relations for short unless confusion occurs. Point relations will be denoted by lower case Roman letters such as $x, y, \cdots$. Throughout this section, we assume that a fuzzy relation algebra $\mathcal{F}$ satisfies axiom P1 of cartesian products.

Definition 4.3.1 A **point relation** x is a crisp relation such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\nabla x = x$ and $x\nabla = \nabla_i$ for some $i \in I$. □

For instance, consider the fuzzy relation algebra $\mathcal{F}^2$ provided in example 4.2.1. If x and y are point relations in $\mathcal{F}$ in the sense of definition 3.3.1, then $\langle x, O\rangle$ is a point relation in the sense of the last definition 4.3.1. Although a relation $\langle x, y\rangle$ in $\mathcal{F}^2$ is a point relation in the sense of definition 3.1.1, that is not a point relation in the sense of definition 4.3.1.

Note that there is a unique $i \in I$ such that $x\nabla = \nabla_i$ for each point relation x by axiom P1.

Let x be a point relation, then $x \sqsubseteq x\nabla = \nabla_i$ for some $i \in I$. Hence the relation x is an element of $\mathcal{F}(\nabla_i)$. Since an algebra $\mathcal{F}(\nabla_i)$ is a fuzzy relation algebra with the identity relation id_i, it holds that $x\mathrm{id}_i = \mathrm{id}_i x = x$. The next proposition shows that point relations (as defined in definition 4.3.1) in $\mathcal{F}$ are in fact point relations in $\mathcal{F}(\nabla_i)$ in the sense of definition 3.3.1.

Proposition 4.3.1 *If x is a point relation such that $x\nabla = \nabla_i$, then the following holds:*

(a) $x \sqcap k\nabla_i = kx$ *for each scalar* k.

(b) $x^\sharp x \sqsubseteq \mathrm{id}_i$.

(c) $\mathrm{id}_i \sqsubseteq xx^\sharp$.

(d) $\nabla_i x = x$.

Proof. (a) Since x and ∇_i are crisp with $x \sqsubseteq x\nabla = \nabla_i$, it is deduced from proposition 3.1.2(d).
(b) Since $x^\sharp x \sqsubseteq (x\nabla)^\sharp(x\nabla) = \nabla_i^\sharp \nabla_i \sqsubseteq \nabla_i$, we have $x^\sharp x \sqsubseteq \mathrm{id} \sqcap \nabla_i = \mathrm{id}_i$ by definition 4.3.1.
(c) follows from $\mathrm{id}_i = \mathrm{id} \sqcap \nabla_i = \mathrm{id} \sqcap x\nabla \sqsubseteq x(x^\sharp \mathrm{id} \sqcap \nabla) = xx^\sharp$.
(d) follows from $\nabla_i x \sqsubseteq \nabla x = x = \mathrm{id}_i x \sqsubseteq \nabla_i x$. □

Let X_i be the set of all point relations x such that $x\nabla = \nabla_i$. Then it is immediate by the last proposition 4.3.1 that X_i is the set of all point relations in $\mathcal{F}(\nabla_i)$ in the sense of definition 3.3.1.

By making use of the definition of point relations in fuzzy relation algebras satisfying axiom P1 of cartesian products we add the following axiom:

Definition 4.3.2 A fuzzy relation algebra $\mathcal{F}$ satisfies the **point axiom for cartesian products** iff:
P2. For each nonzero relation α there are a scalar $k > 0$ and point relations x, y such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ and $x^\sharp y \neq O$. □

For any two point relations according to definition 3.3.1, every pair relation $x^\sharp y$ is nonzero. But there are two point relations according to definition 4.3.1 x and y such that $x\nabla = \nabla_i$ and $y\nabla = \nabla_j$ with $i \neq j$ in a fuzzy relation algebras $\mathcal{F}$ satisfying axiom P1. Then it holds that $x^\sharp y = O$ by proposition 4.2.1 since $x \sqsubseteq \nabla_i$ and $y \sqsubseteq \nabla_j$. For example, consider two point relations $x_1 = \langle x, O\rangle$ and $y_2 = \langle O, y\rangle$ in a fuzzy relation algebra $\mathcal{F}^2$ provided in example 4.2.1, where x and y are point relations in $\mathcal{F}$ in the sense of definition 3.3.1. Then ${x_1}^\sharp y_2 = \langle x^\sharp O, O^\sharp y\rangle = \langle O, O\rangle = O^2$ holds. So compared with the point axiom F4 provided in section 3.3, we need the additional condition $x^\sharp y \neq O$.

Proposition 4.3.2 *Let $\mathcal{F}$ be a fuzzy relation algebra satisfying the point axiom for cartesian products. Then there are a scalar $k > 0$ and point relations $x, y \in X_i$ such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ for each nonzero relation $\alpha \in \mathcal{F}(\nabla_i)$ and each $i \in I$.*

Proof. Let α be an arbitrary nonzero relation in $\mathcal{F}(\nabla_i)$. Then there are a scalar $k > 0$ and point relations x, y such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ and $x^\sharp y \neq O$ by the point axiom P2. It is necessary that the point relations x, y are in a set X_j in order that $x^\sharp y \neq O$ holds. Assume that $j \neq i$, then $\alpha \sqcap x^\sharp y \sqsubseteq \nabla_i \sqcap \nabla_j = O$. This is a contradiction to $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, since $x^\sharp y \neq O$ and $k > 0$. Therefore $x, y \in X_i$. □

In section 3.3 we proved that every relation α in a fuzzy relation algebra satisfying the point axiom F4 can be represented as a supremum of pairs of point relations with semi-scalar weights, and also that such a fuzzy relation algebra is isomorphic to an algebra of all fuzzy relations on a set of all point relations (in the sense of definition 3.3.1). Following section 3.3 we define functions $\chi_i : \mathcal{F}(\nabla_i) \to F\text{-}\mathbf{Rel}(X_i)$ as $\chi_i(\alpha)(x, y) = k$ with $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ for each $\alpha \in \mathcal{F}(\nabla_i)$, each two point relations $x, y \in X_i$ and each $i \in I$, where $F\text{-}\mathbf{Rel}(X_i)$ is a set of all fuzzy relations on the set X_i of all point relations x such that $x\nabla = \nabla_i$. By the last proposition 4.3.2, if $\mathcal{F}$ satisfies the axiom P1 of cartesian products and the point axiom P2 for cartesian products, $\mathcal{F}(\nabla_i)$ is a fuzzy relation algebra satisfying the point axiom F4. Thus every relation $\alpha \in \mathcal{F}(\nabla_i)$ has a unique representation

$$\alpha = \sqcup_{x,y \in X_i} \chi_i(\alpha)(x, y)(x^\sharp y) \ ,$$

and χ_i is an isomorphism from $\mathcal{F}(\nabla_i)$ to $F\text{-}\mathbf{Rel}(X_i)$.

From the above discussion the following two propositions hold.

Corollary 4.3.1 (Representation Theorem) *Let $\mathcal{F}$ be a fuzzy relation algebra satisfying the point axiom for cartesian products. Then every relation α has a unique representation*

$$\alpha = \sqcup_{i\in I} \sqcup_{x,y\in X_i} \chi_i(\phi_i(\alpha))(x,y)(x^\sharp y) \ .$$

Proof. For each $i \in I$ and each relation $\alpha \in \mathcal{F}$, the relation $\phi_i(\alpha)$ has a unique representation $\phi_i(\alpha) = \sqcup_{x,y\in X_i} \chi_i(\phi_i(\alpha))(x,y)(x^\sharp y)$ since $\phi_i(\alpha) \in \mathcal{F}(\nabla_i)$. And also the relation α has a unique representation $\alpha = \sqcup_{i\in I}\phi_i(\alpha)$ by theorem 4.2.1. Thus relation α has such a unique representation. □

Corollary 4.3.2 (Isomorphism Theorem) *Every fuzzy relation algebra $\mathcal{F}$ satisfying the point axiom for cartesian products is isomorphic to the cartesian product $\prod_{i\in I} F\text{-}\mathbf{Rel}(X_i)$ of fuzzy relations $F\text{-}\mathbf{Rel}(X_i)$.*

Proof. The function ψ is an isomorphism from $\mathcal{F}$ to $\prod_{i\in I} \mathcal{F}(\nabla_i)$ and the function χ_i is an isomorphism from $\mathcal{F}(\nabla_i)$ to $F\text{-}\mathbf{Rel}(X_i)$ for each $i \in I$. Therefore the composite function $(\prod_{i\in I} \chi_i) \circ \psi$ is an isomorphism from $\mathcal{F}$ to $\prod_{i\in I} F\text{-}\mathbf{Rel}(X_i)$. □

It is clear that a fuzzy relation algebra $\mathcal{F}$ satisfying the point axiom P2 for cartesian products is isomorphic to (ordinary) fuzzy relations $F\text{-}\mathbf{Rel}(X)$ on a set of all point relations in $\mathcal{F}$ if the set I is a one element set. So we can say that the characterisation in this chapter is natural extension of chapter 3.

Chapter 5

Representation Theorems for Dedekind Categories and Zadeh Categories

A representation theorem for (homogeneous) fuzzy relation algebras satisfying the point axiom has been given in section 3.4. The aim of this chapter is to provide a categorical formalisation for fuzzy relations. Fuzzy relations from a set X to a set Y are set-functions with values in the unit interval $[0,1]$, that is, functions $R : X \times Y \to [0,1]$. The set of all such fuzzy relations from X to Y constitutes a fuzzy relation category. The category of fuzzy relations is naturally equipped with semi-scalar multiplication by scalars in the unit interval. Also, unlike Boolean relation categories, fuzzy relation categories are not Schröder categories [OS80, Jon88]. Instead of Schröder categories we need to adopt Dedekind categories, named by Olivier and Serrato [OS80, OS95], in order to study fuzzy relation categories.

This chapter is organized as follows:

In section 5.1 we show a simple representation theorem for Dedekind categories [OS95] with a unit object satisfying a condition, called the strict point axiom. In section 5.2 we first state the definition of Zadeh categories as a categorical structure formed by fuzzy relations with sup-min composition. Then some useful properties on point relations, due to Schmidt and Ströhlein [SS85], and the point axiom will be introduced to prove a representation theorem for morphisms in Zadeh categories. In particular, the point axiom enables us to construct a function assigning a concrete fuzzy relation between the sets of all point relations to an abstract relation in Zadeh categories. In section 5.3 we show a representation theorem for Zadeh categories satisfying a point axiom without the assumption of existence of unit objects. And it is proved that the representation function is a bijection preserving all operations of Zadeh categories. Therefore the representation function turns out to be a functor. However this functor is not generally an isomorphism of categories since it maps all objects with no points to the empty set. Finally we prove that the representation theorem coincides with the simple representation theorem in section 5.1, when the Zadeh category has a unit object satisfying the strict point axiom.

This chapter is based on [KFM96].

5.1 Simple Representation Theorem for Dedekind Categories

In this section we show a simple proof of a representation theorem for Dedekind categories provided in section 2.3 with a unit object satisfying the strict point axiom.

First we provide some basic properties of Dedekind categories which are necessary to prove the representation theorem.

Proposition 5.1.1 *Let $\alpha, \alpha' : X \rightharpoonup Y$ and $\beta, \beta' : Y \rightharpoonup Z$ be morphisms in $\mathcal{D}$. Then the following holds:*

(a) *If $\alpha \sqcup \alpha' = \nabla_{XY}$, $\alpha \sqcap \alpha' = O_{XY}$ and $\nabla_{XX}\alpha = \alpha$, then $\nabla_{XX}\alpha' = \alpha'$.*

(b) *If $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ and $\alpha'^\sharp\alpha' \sqsubseteq \mathrm{id}_Y$, then $\alpha \sqsubseteq \alpha'$ implies $\alpha = \alpha'$.*

(c) *If* $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}_Y$, *then* $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.

(d) *If* $u \sqsubseteq \mathrm{id}_X$ *and* $v \sqsubseteq \mathrm{id}_X$, *then* $u^\sharp = uu = u$ *and* $uv = u \sqcap v$.

(e) *If* $u \sqsubseteq \mathrm{id}_X$ *and* $v \sqsubseteq \mathrm{id}_Y$, *then* $u\alpha = \alpha \sqcap u\nabla_{XY}$ *and* $\alpha v = \alpha \sqcap \nabla_{XY} v$.

Proof. (a) Note that $\alpha' = \mathrm{id}_X\alpha' \sqsubseteq \nabla_{XX}\alpha'$, and

$$\begin{array}{rcl} \nabla_{XX}\alpha' \sqcap \alpha & \sqsubseteq & \nabla_{XX}(\alpha' \sqcap \nabla^\sharp_{XX}\alpha) \\ & = & \nabla_{XX}(\alpha' \sqcap \nabla_{XX}\alpha) \\ & = & \nabla_{XX}(\alpha' \sqcap \alpha) \\ & = & \nabla_{XX}O_{XY} = O_{XY} \ . \end{array}$$

Hence we have

$$\begin{array}{rcl} \nabla_{XX}\alpha' & = & \nabla_{XX}\alpha' \sqcap \nabla_{XY} \\ & = & \nabla_{XX}\alpha' \sqcap (\alpha \sqcup \alpha') \\ & = & (\nabla_{XX}\alpha' \sqcap \alpha) \sqcup (\nabla_{XX}\alpha' \sqcap \alpha') \\ & = & \alpha' \ . \end{array}$$

(b) follows from $\alpha' \sqsubseteq \mathrm{id}_X\alpha' \sqsubseteq \alpha\alpha^\sharp\alpha' \sqsubseteq \alpha(\alpha')^\sharp\alpha' \sqsubseteq \alpha\mathrm{id}_Y = \alpha$.
(c) It is trivial that $\alpha(\beta \sqcap \beta') \sqsubseteq \alpha\beta \sqcap \alpha\beta'$. Next assume that $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}_Y$, then we have

$$\alpha\beta \sqcap \alpha\beta' \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\alpha\beta') \sqsubseteq \alpha(\beta \sqcap \mathrm{id}_Y\beta') = \alpha(\beta \sqcap \beta')$$

by axiom D3.
(d) Assume that $u \sqsubseteq \mathrm{id}_X$. Then we have

$$u = u \sqcap \mathrm{id}_X \sqsubseteq u(\mathrm{id}_X \sqcap u^\sharp\mathrm{id}_X) \sqsubseteq uu^\sharp \sqsubseteq u^\sharp$$

by axiom D3. Similarly it can be shown that $u^\sharp \sqsubseteq u$ holds. Moreover $uu \sqsubseteq u$ is trivial by proposition 2.3.1(c), and $u = u \sqcap \mathrm{id}_X \sqsubseteq u(\mathrm{id}_X \sqcap u^\sharp\mathrm{id}_X) = uu$ by axiom D3. Assume also that $v \sqsubseteq \mathrm{id}_X$. Then we have

$$uv = uv \sqcap v \sqsubseteq u \sqcap v$$

and

$$u \sqcap v \sqsubseteq u(\mathrm{id} \sqcap u^\sharp v) \sqsubseteq uu^\sharp v = uv$$

since $uv \sqsubseteq v$ holds.
(e) Assume that $u \sqsubseteq \mathrm{id}_X$. Then we have $u\alpha = u\alpha \sqcap \nabla_{XY} \sqsubseteq u(\alpha \sqcap u^\sharp\nabla_{XY}) \sqsubseteq \alpha \sqcap u\nabla_{XY} \sqsubseteq u(u^\sharp\alpha \sqcap \nabla_{XY}) = u\alpha$ by axiom D3, (d) and proposition 2.3.1(c). □

Next we describe properties of morphisms of Dedekind categories which correspond to vectors in [SS85, SS93].

Proposition 5.1.2 *Let* $\alpha : X \rightharpoonup Y$ *be a morphism such that* $\nabla_{XX}\alpha = \alpha$. *Then the following three conditions are equivalent:*

(a) $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$.

(b) $\nabla_{XX} = \alpha\alpha^\sharp$,

(c) $\nabla_{XX} = \alpha\nabla_{YX}$.

Proof. (a)$\Longrightarrow$(b) If $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$, then $\nabla_{XX} = \nabla_{XX}\mathrm{id}_X \sqsubseteq \nabla_{XX}\alpha\alpha^\sharp = \alpha\alpha^\sharp$.
(b)$\Longrightarrow$(c) If $\nabla_{XX} = \alpha\alpha^\sharp$, then $\nabla_{XX} = \alpha\alpha^\sharp \sqsubseteq \alpha\nabla_{YX}$.
(c)$\Longrightarrow$(a) If $\nabla_{XX} = \alpha\nabla_{YX}$, then $\mathrm{id}_X = \mathrm{id}_X \sqcap \nabla_{XX} = \mathrm{id}_X \sqcap \alpha\nabla_{YX} \sqsubseteq \alpha(\alpha^\sharp\mathrm{id}_X \sqcap \nabla_{YX}) = \alpha\alpha^\sharp$. □

We now introduce the three notions of unit objects, I-crisp morphisms and I-points in Dedekind categories. A unit object is an abstraction of the notion of singleton (or one-point) sets. Two distinct univalent and total relations from a unit object are not mutually disjoint. To this end, I-points are in addition assumed to be I-crisp.

Definition 5.1.1 Let $\mathcal{D}$ be a Dedekind category.

(a) An object I in $\mathcal{D}$ is called a **unit object** if $O_{II} \neq \mathrm{id}_I = \nabla_{II}$.

(b) A morphism $\tau : I \rightharpoonup X$ is I-**crisp** if $u\rho \sqsubseteq \tau$ implies $\rho \sqsubseteq \tau$ for a nonzero morphism $u : I \rightharpoonup I$ and a morphism $\rho : I \rightharpoonup X$.

(c) An I-**point** p of X is an I-crisp morphism $p : I \rightharpoonup X$ such that $p^\sharp p \sqsubseteq \mathrm{id}_X$, $\mathrm{id}_I = pp^\sharp$. For each object X, the set of all I-points of X is denoted by $\xi(X)$. □

By making use of the last definition of I-points in Dedekind categories, we add the following axiom:

Definition 5.1.2 A unit object I satisfies the **strict point axiom** iff:
U1. $\sqcup_{p\in\xi(X)} p^\sharp p = \mathrm{id}_X$ for all objects X. □

Note that an I-point p of X is nonzero from its totality condition $\mathrm{id}_I = pp^\sharp$. Thus if an object has at least one I-point then it is nonempty.

Proposition 5.1.3 *Let I be a unit object in $\mathcal{D}$. Then the following holds:*

(a) *If $\tau \sqsubseteq p$ for an I-point $p : I \rightharpoonup X$ and a morphism $\tau : I \rightharpoonup X$, then $\tau = wp$ for a unique morphism $w : I \rightharpoonup I$.*

(b) *If $p_0 \neq p_1$ for I-points $p_0, p_1 : I \rightharpoonup X$, then $p_0 \sqcap p_1 = O_{IX}$ and $p_0 p_1^\sharp = O_{II}$.*

(c) *$(u \sqcap v)\tau = u\tau \sqcap v\tau$ for morphisms $u, v : I \rightharpoonup I$ and $\tau : I \rightharpoonup X$.*

Proof. (a) follows from $\tau p^\sharp p \sqsubseteq \tau = \tau \sqcap p \sqsubseteq (\tau p^\sharp \sqcap \mathrm{id}_I)p = \tau p^\sharp p$ by axiom D3.
(b) First we show that if $p_0 \sqcap p_1 \neq O_{IX}$ then $p_0 = p_1$. As $p_0 \sqcap p_1 \sqsubseteq p_0$, by (a) there is a unique $w : I \rightharpoonup I$ such that $p_0 \sqcap p_1 = wp_0$. If $p_0 \sqcap p_1 \neq O_{IX}$, then $w \neq O_{II}$ and so $p_0 \sqsubseteq p_1$, because $wp_0 \sqsubseteq p_1$ and p_1 is I-crisp. Finally if $p_0 \sqcap p_1 = O_{IX}$, then $p_0 p_1^\sharp = p_0 p_1^\sharp \sqcap \mathrm{id}_I \sqsubseteq (p_0 \sqcap \mathrm{id}_I p_1)p_1^\sharp = (p_0 \sqcap p_1)p_1^\sharp = O_{II}$ by axiom D3.
(c) follows from $(u \sqcap v)\tau \sqsubseteq u\tau \sqcap v\tau \sqsubseteq (u \sqcap v\tau\tau^\sharp)\tau \sqsubseteq (u \sqcap v)\tau$. □

Set $L = \mathcal{D}(I, I)$ and let $\alpha : X \rightharpoonup Y$ be a morphism. Then L is a complete distributive lattice by axiom D1. A function $\xi(\alpha) : \xi(X) \times \xi(Y) \to L$ assigning $\xi(\alpha)(p, q) = p\alpha q^\sharp : I \rightharpoonup I$ to a pair of I-points $p : I \rightharpoonup X$ and $q : I \rightharpoonup Y$, gives an L-relation of $\xi(X)$ into $\xi(Y)$ in the sense of [Gog67]. Now we prove a representation theorem for Dedekind categories with a unit object satisfying the strict point axiom U1, that is every morphism can be represented as a supremum of pairs of I-points with degrees expressed by elements of L.

Theorem 5.1.1 (Representation Theorem) *Let I be a unit object satisfying the strict point axiom. Then every morphism $\alpha : X \rightharpoonup Y$ has a unique representation*

$$\alpha = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp \xi(\alpha)(p, q) q \ ,$$

where $\xi(\alpha)(p, q) = p\alpha q^\sharp$ for I-points $p : I \rightharpoonup X$ and $q : I \rightharpoonup Y$.

Proof. Since $\mathrm{id}_X = \sqcup_{p\in\xi(X)} p^\sharp p$ and $\mathrm{id}_Y = \sqcup_{q\in\xi(Y)} q^\sharp q$ by the strict point axiom U1 we have

$$\begin{aligned} \alpha &= \mathrm{id}_X \alpha \mathrm{id}_Y \\ &= (\sqcup_{p\in\xi(X)} p^\sharp p)\alpha(\sqcup_{q\in\xi(Y)} q^\sharp q) \\ &= \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp p\alpha q^\sharp q \\ &= \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp \xi(\alpha)(p, q) q \ . \end{aligned}$$

Finally we show the uniqueness of the representation. Assume $\alpha = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp k_{p,q} q$. Then for all $p_0 \in \xi(X), q_0 \in \xi(Y)$ we have $\xi(\alpha)(p_0, q_0) = p_0 \alpha q_0^\sharp = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p_0 p^\sharp k_{p,q} q q_0^\sharp = k_{p_0,q_0}$ by proposition 5.1.3(b). □

The following corollary shows that $\xi : \mathcal{D}(X, Z) \to L\text{-}\mathbf{Rel}(\xi(X), \xi(Y))$ is bijective and preserves all operations of L-relations, that is, ξ is a homomorphism of Dedekind categories.

Corollary 5.1.1 *Let I be a unit object satisfying the strict point axiom. Then the following holds:*

(a) *The function $\xi : \mathcal{D}(X,Y) \to L\text{-}\mathbf{Rel}(\xi(X),\xi(Y))$ is bijective.*

(b) $\xi(O_{XY}) = 0_{\xi(X)\xi(Y)}$, $\xi(\nabla_{XY}) = 1_{\xi(X)\xi(Y)}$ *and* $\xi(\mathrm{id}_X) = E_{\xi(X)}$,

(c) $\xi(\alpha \sqcup \alpha') = \xi(\alpha) \cup \xi(\alpha')$ *and* $\xi(\alpha \sqcap \alpha') = \xi(\alpha) \cap \xi(\alpha')$,

(d) $\xi(\alpha^\sharp) = \xi(\alpha)^\cup$,

(e) $\xi(\alpha\beta) = \xi(\alpha)\xi(\beta)$.

Proof. (a) If $\xi(\alpha) = \xi(\beta)$ for morphisms $\alpha, \beta : X \rightharpoonup Y$, then, by the last representation theorem 5.1.1, we have

$$\alpha = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp \xi(\alpha)(p,q) q = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp \xi(\beta)(p,q) q = \beta \ ,$$

which shows that ξ is injective. Given an L-relation $R : \xi(X) \rightharpoonup \xi(Y)$, we set $\alpha_R = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp R(p,q) q$. Then, by the uniqueness of the last representation theorem 5.1.1, we have

$$R(p,q) = \xi(\alpha_R)(p,q) \ ,$$

which shows that ξ is surjective.
(b) It is immediate that $\xi(O_{XY})(p,q) = pO_{XY}q^\sharp = O_{II}$. It follows from $\mathrm{id}_I = \mathrm{id}_I\mathrm{id}_I = pp^\sharp qq^\sharp \sqsubseteq p\nabla_{XY}q^\sharp$ that $\xi(\nabla_{XY})(p,q) = \mathrm{id}_I$, and $\xi(\mathrm{id}_X)(p,q) = p\mathrm{id}_X q^\sharp = pq^\sharp$.
(c) follows from

$$\xi(\alpha \sqcup \alpha')(p,q) = p(\alpha \sqcup \alpha')q^\sharp = p\alpha q^\sharp \sqcup p\alpha' q^\sharp = [\xi(\alpha) \cup \xi(\alpha')](p,q)$$

and

$$\xi(\alpha \sqcap \alpha')(p,q) = \chi(\alpha \sqcap \alpha')q^\sharp = p\alpha q^\sharp \sqcap p\alpha' q^\sharp = [\xi(\alpha) \cap \xi(\alpha')](p,q) \ .$$

(d) follows from $\xi(\alpha^\sharp)(q,p) = q\alpha^\sharp p^\sharp = (p\alpha q^\sharp)^\sharp = p\alpha q^\sharp = \xi(\alpha)(p,q) = \xi(\alpha)^\cup(q,p)$.
(e) follows from $\xi(\alpha\beta)(p,r) = p\alpha\beta r^\sharp = \sqcup_{q\in\xi(Y)} p\alpha q^\sharp q\beta r^\sharp = \sqcup_{q\in\xi(Y)}(p\alpha q^\sharp \sqcap q\beta r^\sharp) = [\xi(\alpha)\xi(\beta)](p,r)$ since $p\alpha q^\sharp, q\beta r^\sharp \sqsubseteq \mathrm{id}_I$. □

From the proof of corollary 5.1.1(b) it is easy to see that $\nabla_{XY} \neq O_{XY}$ for all nonempty objects X and Y (Cf. the condition Z3 in Definition 5.2.3) if $\mathcal{D}$ has a unit object I satisfying the strict point axiom.

Corollary 5.1.2 *Let I be a unit object satisfying the strict point axiom. Then the following holds:*

(a) $\sqcup_{p\in\xi(X)} p = \nabla_{IX}$,

(b) $\nabla_{XY} = \nabla_{XI}\nabla_{IY}$,

(c) *If Y has at least one I-point, then* $\nabla_{XY}\nabla_{YZ} = \nabla_{XZ}$.

Proof. (a) For each I-point $p : I \rightharpoonup X$ we have $p\nabla_{XI} = \nabla_{II} = \mathrm{id}_I$ by proposition 5.1.2. Hence using the strict point axiom U1, $\nabla_{IX} = \nabla_{IX}\mathrm{id}_X = \sqcup_{p\in\xi(X)} \nabla_{IX} p^\sharp p = \sqcup_{p\in\xi(X)} (p\nabla_{XI})^\sharp p = \sqcup_{p\in\xi(X)} p$.
(b) By corollary 5.1.1(b) we have $\xi(\nabla_{XY})(p,q) = \mathrm{id}_I$ and so

$$\nabla_{XY} = \sqcup_{p\in\xi(X)} \sqcup_{q\in\xi(Y)} p^\sharp q = (\sqcup_{p\in\xi(X)} p)^\sharp (\sqcup_{q\in\xi(Y)} q) = \nabla_{IX}^\sharp \nabla_{IY}$$

by the representation theorem 5.1.1 and (a).
(c) Let q be an I-point of Y. Then $\nabla_{IX} = qq^\sharp\nabla_{IX} \sqsubseteq q\nabla_{YX}$ and $\nabla_{IZ} = qq^\sharp\nabla_{IZ} \sqsubseteq q\nabla_{YZ}$. Hence we have

$$\nabla_{XZ} = \nabla_{IX}^\sharp \nabla_{IZ} \sqsubseteq (q\nabla_{YX})^\sharp q\nabla_{YZ} = \nabla_{XY} q^\sharp q \nabla_{YZ} \sqsubseteq \nabla_{XY}\nabla_{YZ}$$

by using (b). □

As a result we have proved that a Dedekind category with a unit object satisfying the strict point axiom is equivalent to a subcategory of a category of L-relations.

5.2 Zadeh Categories

In this section we first state the definition of Zadeh categories as a categorical structure formed by fuzzy relations with sup-min composition. Then some useful properties on point relations, due to Schmidt and Ströhlein [SS85], and a point axiom will be introduced to prove a representation theorem for morphisms in Zadeh categories. In particular, the point axiom induces a function assigning a concrete fuzzy relation between the sets of point relations to an abstract relation in Zadeh categories.

Definition 5.2.1 A **Zadeh category** $\mathcal{Z}$ is a Dedekind category satisfying the following:
Z1. [**Fuzzy Algebra**] For all objects X and Y the homo-set $\mathcal{Z}(X,Y)$ consisting of all morphisms of X into Y is a fuzzy algebra. Its fuzzy algebra structure will be denoted by $\mathcal{Z}(X,Y) = (\mathcal{Z}(X,Y), \sqsubseteq, \sqcup, \sqcap, \cdot, O_{XY}, \nabla_{XY})$.
Z2. [**Compatibility with Semi-Scalar Multiplication**] For morphisms $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$, and a scalar k, the following laws hold:
(a) $k(\alpha\beta) = (k\alpha)(k\beta)$, (b) $(k\alpha)^\sharp = k\alpha^\sharp$, (c) $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla_{YZ})$. □

Throughout the rest of the chapter all discussions will be done in a fixed Zadeh category $\mathcal{Z}$.

Proposition 5.2.1 *Let $\alpha : X \rightharpoonup Y$ and $\beta : Y \rightharpoonup Z$ be morphisms and k a scalar. Then the following holds:*

(a) *$(k\alpha)\beta \sqsubseteq k\nabla_{XZ}$ and $\alpha(k\beta) \sqsubseteq k\nabla_{XZ}$.*

(b) *If β is crisp, then $(k\alpha)\beta = k(\alpha\beta)$. (If α is crisp, then $\alpha(k\beta) = k(\alpha\beta)$).*

(c) *If α and β are crisp, then so are $\alpha^\sharp$ and $\alpha\beta$.*

(d) *The identity morphism id_X is crisp.*

Proof. (a) follows from $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla_{YZ}) \sqsubseteq (k\alpha)(k\nabla_{YZ}) = k(\alpha\nabla_{YZ}) \sqsubseteq k\nabla_{XZ}$ by axioms Z2(c) and Z2(a).
(b) Assume that β is crisp. Then we have

$$(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla_{YZ}) = (k\alpha)(k\beta) = k(\alpha\beta)$$

by axioms Z2(c) and Z2(a).
(c) Assume that α and β are crisp. Then the first follows from

$$\alpha^\sharp \sqcap k\nabla_{YX} = (\alpha \sqcap k\nabla_{XY})^\sharp = (k\alpha)^\sharp = k\alpha^\sharp$$

by propositions 2.3.1(a), 2.3.1(d), and axiom Z2(b); the second follows from

$$\alpha\beta \sqcap k\nabla_{XZ} \sqsubseteq \alpha[\beta \sqcap \alpha^\sharp(k\nabla_{XZ})] \sqsubseteq \alpha(\beta \sqcap k\nabla_{YZ}) = \alpha(k\beta) = k(\alpha\beta)$$

by axiom D3, proposition 3.1.1(a), axiom Z2(c), and (b).
(d) follows from $\mathrm{id}_X \sqcap k\nabla_{XX} = \mathrm{id}_X \sqcap k(\mathrm{id}_X\nabla_{XX}) = \mathrm{id}_X \sqcap (k\mathrm{id}_X)\nabla_{XX} \sqsubseteq (k\mathrm{id}_X)[(k\mathrm{id}_X)^\sharp \mathrm{id}_X \sqcap \nabla_{XX}] = k\mathrm{id}_X$ by axiom D3. □

In view of [KF95, SS85] the concept of points in Zadeh categories is defined as follows:

Definition 5.2.2 A **point** x of an object X is a crisp morphism $x : X \rightharpoonup X$ such that $x^\sharp x \sqsubseteq \mathrm{id}_X$, $\mathrm{id}_X \sqsubseteq xx^\sharp$ and $\nabla_{XX}x = x$. □

Note that a point x of X is automatically nonzero from the totality $\mathrm{id}_X \sqsubseteq xx^\sharp$ if X is nonempty. An empty object has no points because points are nonzero by definition. The set of all points of an object X in a Zadeh category $\mathcal{Z}$ is denoted by $\chi(X)$.

By making use of the last definition of points in Zadeh categories we add the following axiom:

Definition 5.2.3 A Zadeh category $\mathcal{Z}$ satisfies the **point axiom** iff:
Z3. (a) For all objects X and Y the universal morphism ∇_{XY} is nonzero (that is, $\nabla_{XY} \neq O_{XY}$).
(b) For a nonzero morphism $\alpha : X \to Y$ there are a scalar $k > 0$ and points $x : X \to X$ and $y : Y \to Y$ such that $x\alpha \sqcap \nabla_{XY} y = k\nabla_{XY} y$. □

In what follows we assume that the fixed Zadeh category $\mathcal{Z}$ satisfies the point axiom above.

Proposition 5.2.2 *Let $x_0, x_1 : X \to X$, $y, y_0, y_1 : Y \to Y$ points, and k and k' scalars. Then the following holds:*

(a) *If $x_0 \neq x_1$, then $x_0 \sqcap x_1 = O_{XX}$ and $x_0 x_1^\sharp = O_{XX}$.*

(b) *If $\nabla_{XY} y_0 \sqsubseteq \nabla_{XY} y_1$, then $y_0 = y_1$.*

(c) *If $x_0^\sharp \nabla_{XY} y_0 \sqsubseteq x_1^\sharp \nabla_{XY} y_1$, then $x_0 = x_1$ and $y_0 = y_1$.*

(d) *If α is a nonzero crisp morphism and $k\alpha \sqsubseteq k'\alpha'$, then $k \leq k'$.*

(e) *If $\gamma \sqsubseteq \nabla_{XY} y$ and $\nabla_{XX}\gamma = \gamma$, then there is a unique scalar k such that $\gamma = k\nabla_{XY} y$.*

Proof. (a) First we show that if $x_0 \sqcap x_1 \neq O_{XX}$ then $x_0 = x_1$. Assume that $x_0 \sqcap x_1 \neq O_{XX}$. Then by the point axiom there are a scalar $k > 0$ and points z_0, z_1 such that $z_0(x_0 \sqcap x_1) \sqcap z_1 = kz_1$. Hence we have

$$z_1 \sqsubseteq z_0(x_0 \sqcap x_1) \sqsubseteq \nabla_{XX} x_0 \sqcap \nabla_{XX} x_1 = x_0 \sqcap x_1 \ .$$

Therefore $x_0 = z_1 = x_1$ by proposition 5.1.1(b). Finally, if $x_0 \sqcap x_1 = O_{XX}$, then we have

$$x_0 x_1^\sharp = x_0 x_1^\sharp \sqcap \nabla_{XX} \sqsubseteq (x_0 \sqcap \nabla_{XX} x_1) x_1^\sharp = (x_0 \sqcap x_1) x_1^\sharp = O_{XX} \ .$$

(b) Assume that $\nabla_{XY} y \sqsubseteq \nabla_{XY} y_0$. Then $\nabla_{XY} \sqsubseteq \nabla_{XY} y y^\sharp = \nabla_{XY} y_0 y^\sharp$ and so $y_0 y^\sharp \neq O_{YY}$ from $\nabla_{XY} \neq O_{YY}$. Hence $y = y_0$ by (a).
(c) First note that $\nabla_{XY} = \nabla_{XX}^\sharp \nabla_{XY} = (x\nabla_{XX})^\sharp \nabla_{XY} = \nabla_{XX} x^\sharp \nabla_{XY}$ by propositions 5.1.1(a) and 5.1.2. Assume that $x^\sharp \nabla_{XY} y \sqsubseteq x_0^\sharp \nabla_{XY} y_0$. Then we have $\nabla_{XY} y = \nabla_{XX} x^\sharp \nabla_{XY} y \sqsubseteq \nabla_{XX} x_0^\sharp \nabla_{XY} y_0 = \nabla_{XY} y_0$. Hence $y = y_0$ by (b). Similarly $x = x_0$.
(d) By the point axiom there are a scalar $h > 0$ and points x, y such that $x\alpha \sqcap \nabla_{XY} y = h\nabla_{XY} y$. Then we have $h\nabla_{XY} = h\nabla_{XY}\nabla_{YY} = h\nabla_{XY} y \nabla_{YY} \sqsubseteq x\alpha\nabla_{YY}$. But $\nabla_{XY} \sqsubseteq x\alpha\nabla_{YY}$ by proposition 3.1.2(c) since $x\alpha\nabla_{YY}$ is crisp by proposition 5.2.1(c). Hence we have $k\nabla_{XY} \sqsubseteq k\nabla_{XY}\alpha\nabla_{YY} = x(k\alpha)\nabla_{YY} \sqsubseteq x(k'\alpha') \sqsubseteq k'\nabla_{XY}$. Therefore $k \leq k'$ by proposition 3.1.1(e).
(e) It is trivial that if $\gamma = O_{XY}$ then $\gamma = 0\nabla_{XY} y$ by axiom A2(a). Next assume $\gamma \neq 0$. Then by the point axiom there are a scalar $k > 0$ and points x_0, y_0 such that $x_0\gamma \sqcap \nabla_{XY} y_0 = k\nabla_{XY} y_0$. Note that $\gamma = \nabla_{XX}\gamma = x_0\nabla_{XX}\gamma = x_0\gamma$ by proposition 5.1.2. Hence $k\nabla_{XY} y_0 \sqsubseteq x_0\gamma = \gamma \sqsubseteq \nabla_{XY} y$, which implies $y_0 = y$ by (b) and so $\gamma = k\nabla_{XY} y$. The uniqueness of k follows from (d). □

Proposition 5.2.3 *Let $\chi(X)$ be the set of all points of an object X in $\mathcal{Z}$. Then the following holds:*

(a) $\nabla_{XX} = \sqcup_{x \in \chi(X)} x$.

(b) $\mathrm{id}_X = \sqcup_{x \in \chi(X)} x^\sharp x$.

Proof. (a) Set $\alpha = \sqcup_{x \in \chi(X)} x$. It is clear that $\alpha \sqsubseteq \nabla_{XX}$ and $\nabla_{XX}\alpha = \alpha$. As points are crisp by the definition, α is also crisp by proposition 3.1.2(a) and by axiom A3 there is a morphism $\overline{\alpha}$ such that $\alpha \sqcup \overline{\alpha} = \nabla_{XX}$ and $\alpha \sqcap \overline{\alpha} = O_{XX}$. Then $\nabla_{XX}\overline{\alpha} = \overline{\alpha}$ by proposition 5.1.1(a). Assume $\overline{\alpha} \neq O_{XX}$. By the point axiom Z3(b) there are a scalar $k > 0$ and points $x_0, x_1 : X \to X$ such that $x_0\overline{\alpha} \sqcap x_1 = kx_1$. Hence $kx_1 \sqsubseteq x_0\overline{\alpha} \sqsubseteq \nabla_{XX}\overline{\alpha} = \overline{\alpha}$. But $\overline{\alpha}$ is crisp by proposition 3.1.2(b) and so $x_1 \sqsubseteq \overline{\alpha}$ by proposition 3.1.2(c). Therefore $x_1 \sqsubseteq \alpha \sqcap \overline{\alpha} = O_{XX}$, which

contradicts $x_1 \neq O_{XX}$. This proves $\overline{\alpha} = O_{XX}$.
(b) It is clear that $\sqcup_{x\in\chi(X)} x^\sharp x \sqsubseteq \mathrm{id}_X$. Also we have

$$\mathrm{id}_X = \mathrm{id}_X \sqcap \nabla_{XX} = \sqcup_{x\in\chi(X)}(\mathrm{id}_X \sqcap x) \sqsubseteq \sqcup_{x\in\chi(X)}(\mathrm{id}_X x^\sharp \sqcap \mathrm{id}_X)x \sqsubseteq \sqcup_{x\in\chi(X)} x^\sharp x$$

by using (a). □

Thus from the point axiom Z3(b) and proposition 5.2.2(e) there exists a unique scalar $\chi(\alpha)(x,y)$ such that

$$x\alpha \sqcap \nabla_{XY} y = \chi(\alpha)(x,y)\nabla_{XY} y \ .$$

Thus $\chi(\alpha)$ defines a fuzzy relation from the set of all points of X into the set of all points of Y. In the next section we will prove that the function $\chi : \mathcal{Z} \to F\text{-}\mathbf{Rel}$ is an embedding of Zadeh categories.

5.3 Representation Theorem for Zadeh Categories

First we prove the representation theorem for Zadeh categories asserting that every morphism in a Zadeh category satisfying the point axiom can be represented as a union of pairs of points with semi-scalar weights.

Theorem 5.3.1 (Representation Theorem) *Every morphism $\alpha : X \rightharpoonup Y$ in a Zadeh category $\mathcal{Z}$ satisfying the point axiom has a unique representation*

$$\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y \ ,$$

where $\chi(\alpha)(x,y)$ is a unique scalar such that $x\alpha \sqcap \nabla_{XY} y = \chi(\alpha)(x,y)\nabla_{XY} y$.

Proof. Since $\nabla_{YY} = \sqcup_{y\in\chi(Y)} y$ by proposition 5.2.3(a) we have

$$\begin{array}{rcl} x\alpha & = & x\alpha \sqcap \nabla_{XY} \\ & = & x\alpha \sqcap \nabla_{XY}\nabla_{YY} \\ & = & x\alpha \sqcap \nabla_{XY}(\sqcup_{y\in\chi(Y)} y) \\ & = & \sqcup_{y\in\chi(Y)}(x\alpha \sqcap \nabla_{XY} y) \\ & = & \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y)\nabla_{XY} y \end{array}$$

and so

$$\begin{array}{rcl} \alpha & = & \mathrm{id}_X \alpha \\ & = & \sqcup_{x\in\chi(X)} x^\sharp x\alpha \\ & = & \sqcup_{x\in\chi(X)} x^\sharp [\sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y)\nabla_{XY} y] \\ & = & \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp [\chi(\alpha)(x,y)\nabla_{XY} y] \\ & = & \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y \end{array}$$

using $\mathrm{id}_X = \sqcup_{x\in\chi(X)} x^\sharp x$ by proposition 5.2.3(b). Finally we show the uniqueness of the representation. Assume $\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y$. Then for all $x_0 \in \chi(X), y_0 \in \chi(Y)$ we have

$$x_0\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} k_{x,y} x_0 x^\sharp \nabla_{XY} y = \sqcup_{y\in\chi(Y)} k_{x_0,y} \nabla_{XY} y$$

and

$$x_0\alpha \sqcap \nabla_{XY} y_0 = \sqcup_{y\in\chi(Y)} [k_{x_0,y}\nabla_{XY} y \sqcap \nabla_{XY} y_0] = k_{x_0,y_0}\nabla_{XY} y_0 \ .$$

Hence $k_{x,y} = \chi(\alpha)(x,y)$ by proposition 5.2.2(e). □

Corollary 5.3.1 *The function $\chi : \mathcal{Z}(X,Y) \to F\text{-}\mathbf{Rel}(\chi(X),\chi(Y))$ is a bijection for all objects X and Y in a Zadeh category $\mathcal{Z}$.*

Proof. If $\chi(\alpha) = \chi(\alpha')$ for morphisms $\alpha, \alpha' : X \rightharpoonup Y$, then by the last theorem we have

$$\begin{array}{rcl} \alpha & = & \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y \\ & = & \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha')(x,y) x^\sharp \nabla_{XY} y \\ & = & \alpha' \ , \end{array}$$

which shows that χ is injective. Given a fuzzy relation $R : \chi(X) \times \chi(Y) \to [0,1]$, we set

$$\alpha_R = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} R(x,y) x^\sharp \nabla_{XY} y \ .$$

Then by the uniqueness of representation in the last theorem we have $R(x,y) = \chi(\alpha_R)(x,y)$, which means that χ is surjective. □

Theorem 5.3.2 *For a morphism $\alpha : X \multimap Y$ and a set $\{k_\lambda\}_\lambda$ of scalars the identity $\sqcup_\lambda k_\lambda \alpha = (\vee_\lambda k_\lambda)\alpha$ holds.*

Proof. First we show that $\sqcup_\lambda k_\lambda x^\sharp \nabla_{XY} y = (\vee_\lambda k_\lambda) x^\sharp \nabla_{XY} y$ for points $x : X \multimap X$ and $y : Y \multimap Y$. It is obvious that $\sqcup_\lambda k_\lambda x^\sharp \nabla_{XY} y \sqsubseteq (\vee_\lambda k_\lambda) x^\sharp \nabla_{XY} y$ and so there is a scalar k such that $\sqcup_\lambda k_\lambda (x^\sharp \nabla_{XY} y) = k x^\sharp \nabla_{XY} y$. Then $k \le \vee_\lambda k_\lambda$ by $k x^\sharp \nabla_{XY} y \sqsubseteq (\vee_\lambda k_\lambda) x^\sharp \nabla_{XY} y$ and proposition 5.2.2(b). On the other hand $k_\lambda \le k$ from $k_\lambda x^\sharp \nabla_{XY} y \sqsubseteq k x^\sharp \nabla_{XY} y$ and so $\vee_\lambda k_\lambda \le k$. Therefore this proves $k = \vee_\lambda k_\lambda$.

We are now ready to prove the general case. Since $\alpha = \sqcup_{x,y\in X} \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y$ by the representation theorem 5.3.1 we have

$$\begin{aligned}
\sqcup_\lambda k_\lambda \alpha &= \sqcup_\lambda k_\lambda [\sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y] \\
&= \sqcup_\lambda \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} [k_\lambda \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y] \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \sqcup_\lambda [k_\lambda \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y] \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} [(\vee_\lambda k_\lambda) \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y] \\
&= (\vee_\lambda k_\lambda) [\sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) x^\sharp \nabla_{XY} y] \\
&= (\vee_\lambda k_\lambda)\alpha \ .
\end{aligned}$$

□

The following theorem corresponds to the Tarski rule for Boolean relation algebras [SS85, Tar41], see also section 3.4.

Theorem 5.3.3 *For a nonzero morphism $\alpha : X \multimap Y$ in $\mathcal{Z}$ there is a scalar $k > 0$ such that $\nabla_{XX} \alpha \nabla_{YY} = k \nabla_{XY}$.*

Proof. By means of the representation theorem 5.3.1 we have

$$\begin{aligned}
\nabla_{XX} \alpha \nabla_{YY} &= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \nabla_{XX} [\chi(\alpha)(x,y) x^\sharp \nabla_{XY} y] \nabla_{YY} \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) [\nabla_{XX} x^\sharp \nabla_{XY} y \nabla_{YY}] \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) \nabla_{XX} \nabla_{XY} \nabla_{YY} \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \chi(\alpha)(x,y) \nabla_{XY} \\
&= [\vee_{x\in\chi(X)} \vee_{y\in\chi(Y)} \chi(\alpha)(x,y)] \nabla_{XY} \ .
\end{aligned}$$

Set $k = \vee_{x\in\chi(X)} \vee_{y\in\chi(Y)} \chi(\alpha)(x,y)$. Then $\nabla_{XX} \alpha \nabla_{YY} = k \nabla_{XY}$. There exist points x, y such that $\chi(\alpha)(x,y) > 0$ by the point axiom Z3(b). Hence $k > 0$, which completes the proof. □

The following proposition shows that $\chi : \mathcal{Z} \to F\text{-}\mathbf{Rel}$ preserves all operations of fuzzy morphisms, that is, χ is a homomorphism of Zadeh categories.

Proposition 5.3.1 *Let $\alpha, \alpha' : X \multimap Y, \beta : Y \multimap Z$ be morphisms and k a scalar. Then the following holds:*

(a) $\chi(O_{XY}) = 0_{\chi(X)\chi(Y)}$, $\chi(\nabla_{XY}) = 1_{\chi(X)\chi(Y)}$ *and* $\chi(\mathrm{id}_X) = E_{\chi(X)}$.

(b) $\chi(\alpha \sqcup \alpha') = \chi(\alpha) \cup \chi(\alpha')$.

(c) $\chi(\alpha \sqcap \alpha') = \chi(\alpha) \cap \chi(\alpha')$.

(d) $\chi(\alpha^\sharp) = \chi(\alpha)^\cup$.

(e) $\chi(k\alpha) = k\chi(\alpha)$.

(f) $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$.

Proof. (a) The first follows from $xO_{XY} \sqcap \nabla_{XY} y = O_{XY} = 0y$, the second follows from $x\nabla_{XY} \sqcap \nabla_{XY} y = \nabla_{XY} \sqcap \nabla_{XY} y = \nabla_{XY} y$ by $x\nabla_{XY} = x\nabla_{XX}\nabla_{XY} = \nabla_{XX}\nabla_{XY} = \nabla_{XY}$, and the last follows from $x\mathrm{id}_X \sqcap \nabla_{XX} x' = x \sqcap x'$ and proposition 5.2.2(e).

(b) follows from

$$\begin{aligned}
\chi(\alpha \sqcup \beta)(x,y)\nabla_{XY} y &= x(\alpha \sqcup \beta) \sqcap \nabla_{XY} y \\
&= (x\alpha \sqcup x\beta) \sqcap \nabla_{XY} y \\
&= (x\alpha \sqcap \nabla_{XY} y) \sqcup (x\beta \sqcap \nabla_{XY} y) \\
&= \chi(\alpha)(x,y)\nabla_{XY} y \sqcup \chi(\beta)(x,y)\nabla_{XY} y \\
&= [\chi(\alpha)(x,y) \vee \chi(\beta)(x,y)]\nabla_{XY} y \\
&= [\chi(\alpha) \cup \chi(\beta)](x,y)\nabla_{XY} y \ .
\end{aligned}$$

(c) First note that $x(\alpha \sqcap \alpha') = x\alpha \sqcap x\alpha'$. It follows from $x^\sharp x \sqsubseteq \mathrm{id}_X$ that

$$x(\alpha \sqcap \alpha') \sqsubseteq x\alpha \sqcap x\alpha' \sqsubseteq x(\alpha \sqcap x^\sharp x\alpha') \sqsubseteq x(\alpha \sqcap \alpha') \ .$$

Hence we have

$$\begin{aligned}
\chi(\alpha \sqcap \alpha')(x,y)\nabla_{XY} y &= x(\alpha \sqcap \alpha') \sqcap \nabla_{XY} y \\
&= (x\alpha \sqcap x\alpha') \sqcap \nabla_{XY} y \\
&= (x\alpha \sqcap \nabla_{XY} y) \sqcap (x\alpha' \sqcap \nabla_{XY} y) \\
&= \chi(\alpha)(x,y)\nabla_{XY} y \sqcap \chi(\alpha')(x,y)\nabla_{XY} y \\
&= [\chi(\alpha)(x,y) \wedge \chi(\alpha')(x,y)]\nabla_{XY} y \\
&= [\chi(\alpha) \cap \chi(\alpha')](x,y)\nabla_{XY} y \ .
\end{aligned}$$

(d) First we show that $\alpha \sqcap x^\sharp \nabla_{XY} y = kx^\sharp \nabla_{XY} y$ if and only if $x\alpha \sqcap \nabla_{XY} y = k\nabla_{XY} y$. Note that

$$x\alpha \sqcap \nabla_{XY} y = x(\alpha \sqcap x^\sharp \nabla_{XY} y)$$

and

$$\alpha \sqcap x^\sharp \nabla_{XY} y = x^\sharp (x\alpha \sqcap \nabla_{XY} y) \ ,$$

because

$$\begin{aligned}
x\alpha \sqcap \nabla_{XY} y &\sqsubseteq x(\alpha \sqcap x^\sharp \nabla_{XY} y) \\
&\sqsubseteq xx^\sharp (x\alpha \sqcap \nabla_{XY} y) \\
&\sqsubseteq \nabla(x\alpha \sqcap \nabla_{XY} y) \\
&\sqsubseteq x\alpha \sqcap \nabla_{XY} y \ ,
\end{aligned}$$

and

$$\alpha \sqcap x^\sharp \nabla_{XY} y \sqsubseteq x^\sharp (x\alpha \sqcap \nabla_{XY} y) \sqsubseteq x^\sharp x(\alpha \sqcap x^\sharp \nabla_{XY} y) \sqsubseteq \alpha \sqcap x^\sharp \nabla_{XY} y$$

by $x^\sharp x \sqsubseteq \mathrm{id}_X$. Now assume $\alpha \sqcap x^\sharp \nabla_{XY} y = kx^\sharp \nabla_{XY} y$. Then we have

$$x\alpha \sqcap \nabla_{XY} y = x(\alpha \sqcap x^\sharp \nabla_{XY} y) = x[k(x^\sharp \nabla_{XY} y)] = k[x(x^\sharp \nabla_{XY} y)] = k\nabla_{XY} y \ .$$

Conversely assume $x\alpha \sqcap \nabla_{XY} y = k\nabla_{XY} y$. Then we have

$$\alpha \sqcap x^\sharp \nabla_{XY} y = x^\sharp (x\alpha \sqcap \nabla_{XY} y) = x^\sharp (k\nabla_{XY} y) = kx^\sharp \nabla_{XY} y$$

since $x^\sharp$ is crisp. An identity $\chi(\alpha^\sharp) = \chi(\alpha)^\cup$ follows from

$$\begin{aligned}
x\alpha \sqcap \nabla_{XY} y = k\nabla_{XY} y &\iff \alpha \sqcap x^\sharp \nabla_{XY} y = kx^\sharp \nabla_{XY} y \\
&\iff \alpha^\sharp \sqcap y^\sharp \nabla_{YX} x = ky^\sharp \nabla_{YX} x \\
&\iff y\alpha^\sharp \sqcap \nabla_{YX} x = k\nabla_{YX} x \ .
\end{aligned}$$

(e) follows from

$$\begin{aligned}
\chi(k\alpha)(x,y)\nabla_{XY} y &= x(k\alpha) \sqcap \nabla_{XY} y \\
&= k(x\alpha) \sqcap k\nabla_{XY} \sqcap \nabla_{XY} y \quad (\ k(x\alpha) \sqsubseteq k\nabla_{XY}\ ) \\
&= k(x\alpha) \sqcap k\nabla_{XY} y \\
&= k(x\alpha \sqcap \nabla_{XY} y) \\
&= k[\chi(\alpha)(x,y)\nabla_{XY} y] \\
&= [k\chi(\alpha)(x,y)]\nabla_{XY} y \\
&= [k\chi(\alpha)](x,y)\nabla_{XY} y \ .
\end{aligned}$$

(f) When $\nabla_{ZZ} = O_{ZZ}$, it is trivial that the equality $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$. Therefore we may assume that Z is nonempty. First note that $\chi(\alpha\beta)(x,z) = \vee_y \chi(\alpha y^\sharp y\beta)(x,z)$ from

$$\begin{aligned}
\chi(\alpha\beta)(x,z)\nabla_{XZ}z &= x\alpha\beta \sqcap \nabla_{XZ}z \\
&= x\alpha \mathrm{id}_Y \beta \sqcap \nabla_{XZ}z \\
&= x\alpha(\sqcup_y y^\sharp y)\beta \sqcap \nabla_{XZ}z \\
&= \sqcup_y (x\alpha y^\sharp y\beta \sqcap \nabla_{XZ}z) \\
&= \sqcup_y \chi(\alpha y^\sharp y\beta)(x,z)\nabla_{XZ}z \\
&= [\vee_y \chi(\alpha y^\sharp y\beta)(x,z)]\nabla_{XZ}z
\end{aligned}$$

by theorem 5.3.2. To complete the proof it suffices to show

$$\chi(\alpha y^\sharp y\beta)(x,z) = \chi(\alpha)(x,y) \wedge \chi(\beta)(y,z) \ .$$

Note that $x\alpha \sqcap \nabla_{XY}y = x\alpha y^\sharp y$ from

$$x\alpha y^\sharp y \sqsubseteq x\alpha \sqcap \nabla_{XY}y \sqsubseteq (x\alpha y^\sharp \sqcap \nabla_{XY})y = x\alpha y^\sharp y \ ,$$

and $\nabla_{XZ}(z\beta^\sharp \sqcap \nabla_{ZY}y) = \nabla_{XZ}z\beta^\sharp \sqcap \nabla_{XY}y$ from

$$\begin{aligned}
\nabla_{XZ}(z\beta^\sharp \sqcap \nabla_{ZY}y) &\sqsubseteq \nabla_{XZ}z\beta^\sharp \sqcap \nabla_{XZ}\nabla_{ZY}y \\
&\sqsubseteq \nabla_{XZ}z\beta^\sharp \sqcap \nabla_{XY}y \\
&\sqsubseteq \nabla_{XZ}(z\beta^\sharp \sqcap \nabla_{ZX}\nabla_{XY}y) \\
&\sqsubseteq \nabla_{XZ}(z\beta^\sharp \sqcap \nabla_{ZY}y) \ .
\end{aligned}$$

Then we have

$$\begin{aligned}
[\chi(\alpha)(x,z) \wedge \chi(\beta)(y,z)]\nabla_{XY}y &= \chi(\alpha)(x,y)\nabla_{XY}y \sqcap \chi(\beta)(y,z)\nabla_{XY}y \\
&= \chi(\alpha)(x,y)\nabla_{XY}y \sqcap \chi(\beta^\sharp)(y,z)\nabla_{XZ}\nabla_{ZY}y \\
&= (x\alpha \sqcap \nabla_{XY}y) \sqcap \nabla_{XZ}(z\beta^\sharp \sqcap \nabla_{ZY}y) \\
&= (x\alpha \sqcap \nabla_{XY}y) \sqcap (\nabla_{XZ}z\beta^\sharp \sqcap \nabla_{XY}y) \\
&= x\alpha y^\sharp y \sqcap \nabla_{XZ}z\beta^\sharp \ .
\end{aligned}$$

Set $r = \chi(\alpha y^\sharp y\beta)(x,z)$ and $s = \chi(\alpha)(x,y) \wedge \chi(\beta)(y,z)$. Then $x\alpha y^\sharp y\beta \sqcap \nabla_{XZ}z = r\nabla_{XZ}z$ and $x\alpha y^\sharp y \sqcap \nabla_{XZ}z\beta^\sharp = s\nabla_{XY}y$. We have $r \leq s$ from

$$r\nabla_{XZ}z = x\alpha y^\sharp y\beta \sqcap \nabla_{XZ}z \sqsubseteq (x\alpha y^\sharp y \sqcap \nabla_{XZ}z\beta^\sharp)\beta \sqsubseteq (s\nabla_{XY}y)\beta \sqsubseteq s\nabla_{XZ} \ ,$$

and $s \leq r$ from

$$s\nabla_{XY}y = x\alpha y^\sharp y \sqcap \nabla_{XZ}z\beta^\sharp \sqsubseteq (x\alpha z^\sharp z\beta \sqcap \nabla_{XZ}z)\beta^\sharp \sqsubseteq (r\nabla_{XZ}z)\beta^\sharp \sqsubseteq r\nabla_{XY} \ .$$

Therefore $r = s$. □

The following is immediately deduced from the last proposition.

Corollary 5.3.2 (Insertion Theorem) *Let $\alpha : X \rightharpoonup Y$ and $\beta : Y \rightharpoonup Z$ be morphisms and let $x : X \rightharpoonup X$ and $z : Z \rightharpoonup Z$ be points. If $\chi(\alpha\beta)(x,z) > 0$, then there is a point $y : Y \rightharpoonup Y$ such that $\chi(\alpha)(x,y) > 0$ and $\chi(\beta)(y,z) > 0$.* □

Proposition 5.3.2 *Let $\mathcal{Z}$ be a Zadeh category satisfying the point axiom and I a unit object in $\mathcal{Z}$. Then the following holds:*

(a) *If $u : I \rightharpoonup I$ is a morphism, then there is a scalar k such that $u = k\mathrm{id}_I$.*

(b) *A morphism $\tau : I \rightharpoonup X$ is crisp if and only if it is I-crisp.*

(c) *The unit object I satisfies the strict point axiom.*

(d) *$\xi(\alpha)(\nabla_{IX}x, \nabla_{IY}y) = \chi(\alpha)(x,y)\mathrm{id}_I$ for I-points x of X and y of Y.*

Proof. (a) Recall that $\mathrm{id}_I = \nabla_{II}$ and a unit object I has just one point id_I. If u is nonzero, then by the point axiom Z3(a) we have $u \sqcap \mathrm{id}_I = k\mathrm{id}_I$ for a scalar $k > 0$.
(b) First we show that a crisp morphism $\tau : I \rightharpoonup X$ is I-crisp. Assume that τ is crisp and $u\tau' \sqsubseteq \tau$ for $\tau' : I \rightharpoonup X$ and $u \neq O_{II} : I \rightharpoonup I$. By (a) there is a scalar $k > 0$ such that $u = k\mathrm{id}_I$ and so $k\tau' = k(\mathrm{id}_I\tau') = (k\mathrm{id}_I)(k\tau') \sqsubseteq u\tau' \sqsubseteq \tau$. As τ is crisp we have $\tau' \sqsubseteq \tau$ by proposition 3.1.2(c), which indicates that τ is I-crisp. Conversely we show that an I-crisp morphism is crisp. Assume that $\tau : I \rightharpoonup X$ is I-crisp. Note that $\tau = \tau\mathrm{id}_X = \sqcup_{x\in\chi(X)}\tau x^\sharp x$ by proposition 5.2.3(b) If $u = \tau x^\sharp$ is nonzero, then $x \sqsubseteq \tau$ because τ is I-crisp, and so $u = \mathrm{id}_I$. Hence $\tau = \sqcup_{x\sqsubseteq\tau} x$ and consequently τ is crisp by proposition 3.1.2(a).
(c) Let $x : X \rightharpoonup X$ be a point of X and set $p = \nabla_{IX}x$. Then

$$p^\sharp p = x^\sharp \nabla_{XI}\nabla_{IX}x \sqsubseteq x^\sharp \nabla_{XX}x = x^\sharp x \sqsubseteq \mathrm{id}_X \ ,$$

and

$$\mathrm{id}_X = \nabla_{II} = \nabla_{IX}\nabla_{XI} \sqsubseteq \nabla_{IX}xx^\sharp\nabla_{XI} = pp^\sharp$$

since X has a point x. As x and ∇_{IX} are crisp, p is crisp by proposition 5.2.1(c) and so p is I-crisp by (b). This proves that $p = \nabla_{IX}x$ is an I-point of X. Hence we have

$$\begin{aligned}
\nabla_{IX} &= \nabla_{IX}\mathrm{id}_X \\
&= \nabla_{IX}(\sqcup_{x\in\chi(X)}x^\sharp x) \\
&= \sqcup_{x\in\chi(X)}\nabla_{IX}x^\sharp x \\
&\sqsubseteq \sqcup_{x\in\chi(X)}\nabla_{IX}x \\
&\sqsubseteq \sqcup_{p\in\xi(X)}p \ ,
\end{aligned}$$

which means $\nabla_{IX} = \sqcup_{p\in\xi(X)}p$.
(d) Set $k = \chi(\alpha)(x, y)$. As has been seen in (c) $\nabla_{IX}x$ and $\nabla_{IY}y$ are I-points and so we have

$$\begin{aligned}
\xi(\alpha)(\nabla_{IX}x, \nabla_{IY}y) &= \nabla_{IX}x\alpha(\nabla_{IY}y)^\sharp \\
&= \nabla_{IX}(x\alpha y^\sharp \sqcap \nabla_{XY})\nabla_{YI} \\
&= \nabla_{IX}(x\alpha y^\sharp \sqcap \nabla_{XY}yy^\sharp)\nabla_{YI} \\
&= \nabla_{IX}(x\alpha \sqcap \nabla_{XY}y)y^\sharp\nabla_{YI} \\
&= \nabla_{IX}(k\nabla_{XY}y)y^\sharp\nabla_{YI} \\
&= k\nabla_{IX}\nabla_{XY}yy^\sharp\nabla_{YI} \\
&= k\nabla_{IX}\nabla_{XY}\nabla_{YI} \\
&= k\mathrm{id}_I \ .
\end{aligned}$$

□

Consequently we have proved that a Zadeh category satisfying the point axiom is equivalent to a subcategory of the category of sets and (ordinary) fuzzy relations.

Chapter 6

A Representation Theorem for Relation Algebras

In this chapter we consider relation algebras, which may not be Boolean, and provide their representation theorem. Relation algebras in the sense of this chapter are equivalent to Dedekind categories [Gog67] (or allegories [FS90]) with just one object. Section 5.1 proved a representation theorem for Dedekind categories, showing that a Dedekind category with a unit object satisfying the strict point axiom U1 is equivalent to a subcategory of the category of L-relations (where L is the lattice of all endomorphisms on the unit object). A unit object is an abstraction of singleton (or one-point) sets, and, following [Gog67], L-relations in section 2.2 are set-functions with values on a fixed complete distributive lattice L, that is, functions $R : X \times Y \to L$. The discussion in this chapter does not assume the existence of a unit object, and L-relations in this chapter are homogeneous relations on a set X, that is, functions $R : X \times X \to L$. This study is the first step to prove a representation theorem for Dedekind categories without unit objects.

To prove a representation theorem for relation algebras, we use concepts of scalar relations and point relations. The concept of scalar relations is an original one, which is defined in section 6.1 as a relation included in the identity relation and which commutes with the greatest relation with respect to composition. In the case of L-relations, scalar relations can be represented as scalar matrices. We use the concept of scalar relations to define a new concept of crisp relations which is called s-crisp different from that in [KF95, KFM96, Fur97a]. Also the set of all scalar relations is a complete distributive lattice, which is a sublattice of the relation algebra, and scalar relations represent membership values. The concept of point relations was introduced by Schmidt and Ströhlein in [SS85, SS93] in the context of applications of Boolean relation algebras to theories of graphs and programs, and it played an important rôle in proofs of representation theorems in [SS85, KF95, KFM96].

Section 6.1 provides definitions and some properties of scalar relations and s-crisp relations. In section 6.2 we define a "strict" point axiom by using our concepts of scalar relations and point relations. In section 6.3 we prove our representation theorem for relation algebras.

This chapter is based on [Fur97b].

6.1 Scalar Relations

In this section we study a concept of scalar relations in a relation algebra $\mathcal{R}$. Note that relation algebras in this thesis, as defined in section 2.3, are not Boolean.

All elements of the relation algebra $\mathcal{R}$ are called "relations" for short. A relation α is nonzero if $\alpha \neq O$.

First we provide some properties of relation algebras.

Proposition 6.1.1 *Let α, β, β' be relations. Then the following hold:*

(a) *If $\alpha^\sharp \alpha \sqsubseteq \mathrm{id}$, then $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.*

(b) *If $\alpha \sqsubseteq \mathrm{id}$ and $\beta \sqsubseteq \mathrm{id}$, then $\alpha^\sharp = \alpha\alpha = \alpha$ and $\alpha\beta = \alpha \sqcap \beta$.*

(c) *If* $\beta \sqsubseteq \mathrm{id}$ *and* $\beta' \sqsubseteq \mathrm{id}$, *then* $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.

Proof. (a) If $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}$, then $\alpha\beta \sqcap \alpha\beta' \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\alpha\beta') \sqsubseteq \alpha(\beta \sqcap \mathrm{id}\beta') = \alpha(\beta \sqcap \beta')$ by axiom R3.
(b) Assume that $\alpha \sqsubseteq \mathrm{id}$ and $\beta \sqsubseteq \mathrm{id}$. Then we have

$$\alpha = \alpha \sqcap \mathrm{id} \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp \mathrm{id}) \sqsubseteq \mathrm{id} \sqcap \alpha^\sharp \mathrm{id} \sqsubseteq \alpha^\sharp$$

by axiom R3. Similarly it can be shown that $\alpha^\sharp \sqsubseteq \alpha$ holds. Also $\alpha\alpha \sqsubseteq \alpha$ is trivial, and it holds that

$$\alpha = \alpha \sqcap \nabla \sqsubseteq \alpha(\alpha \sqcap \alpha^\sharp \nabla) \sqsubseteq \alpha\alpha$$

by axiom R3. Moreover, since $\alpha\beta \sqsubseteq \beta$, it holds that

$$\alpha\beta = \alpha\beta \sqcap \beta \sqsubseteq \alpha \sqcap \beta \quad \text{and} \quad \alpha \sqcap \beta \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp\beta) \sqsubseteq \alpha\beta$$

by axiom R3.
(c) If $\beta \sqsubseteq \mathrm{id}$ and $\beta' \sqsubseteq \mathrm{id}$, then we have $\alpha\beta \sqcap \alpha\beta' \sqsubseteq (\alpha \sqcap \alpha\beta'\beta^\sharp)\beta \sqsubseteq \alpha\beta'\beta = \alpha(\beta \sqcap \beta')$ by axiom R3 and (b). □

Note that $\alpha(\sqcap_\lambda \beta_\lambda) \sqsubseteq \sqcap_\lambda(\alpha\beta_\lambda)$ and $\nabla\nabla = \nabla$ hold immediately by proposition 2.3.1(c).

The concepts of scalar relations and s-crisp relations in relation algebras are defined by the following:

Definition 6.1.1 Let $\mathcal{R}$ be a relation algebra.

(a) A relation k is called **scalar** if and only if $k \sqsubseteq \mathrm{id}$ and $k\nabla = \nabla k$.

(b) A relation α is called **s-crisp** (scalar crisp) if for all nonzero scalar relations k and all relations β, $k\beta \sqsubseteq \alpha$ implies $\beta \sqsubseteq \alpha$. □

It is trivial that O and id are scalar relations, and that ∇ is s-crisp (but O and id are not necessarily s-crisp).

The concept of I-crisp relations has been defined in section 5.1 on the assumption of the existence of a unit object. The concept of crispness can also be found in section 3.1, where it is defined via semi-scalar multiplication. In this chapter we need neither a unit object, nor semi-scalar multiplication. Instead we used the concept of scalar relations to define s-crisp relations.

Next we provide some basic properties of scalar relations and s-crisp relations.

Proposition 6.1.2 *Let* k *be a scalar relation and* α, β *relations. Then the following holds:*

(a) $k\alpha = \alpha \sqcap k\nabla$ *and* $\alpha k = \alpha \sqcap \nabla k$. *In particular,* $k = \mathrm{id} \sqcap k\nabla$.

(b) $k\alpha = \alpha k$.

(c) $(k \sqcap k')\alpha = \alpha(k \sqcap k')$, $(k \sqcup k')\alpha = \alpha(k \sqcup k')$.

(d) *If* $k\nabla \sqsubseteq k'\nabla$, *then* $k \sqsubseteq k'$.

(e) *If* α *and* β *are s-crisp, then so is* $\alpha \sqcap \beta$.

Proof. (a) Since $k \sqsubseteq \mathrm{id}$ and $\alpha \sqsubseteq \nabla$, it holds that $k\alpha \sqsubseteq \alpha \sqcap k\nabla = k(k^\sharp\alpha \sqcap \nabla) = kk^\sharp\alpha = k\alpha$ by axion R3 and proposition 6.1.1(b). Similarly it can be shown that $\alpha k = \alpha \sqcap \nabla k$.
(b) From (a) it holds that $k\alpha = \alpha \sqcap k\nabla = \alpha \sqcap \nabla k = \alpha k$.
(c) It follows from $(k \sqcap k')\alpha = (kk')\alpha = \alpha(kk') = \alpha(k \sqcap k')$ by proposition 6.1.1(b), and (b). Also it follows from $(k \sqcup k')\alpha = k\alpha \sqcup k'\alpha = \alpha k \sqcup \alpha k' = \alpha(k \sqcup k')$ by proposition 2.3.1(b), and (b).
(d) Assume that $k\nabla \sqsubseteq k'\nabla$. Then $k = \mathrm{id} \sqcap k\nabla \sqsubseteq \mathrm{id} \sqcap k'\nabla = k'$ by (a).
(e) If $k\gamma \sqsubseteq \alpha \sqcap \beta$, then $k\gamma \sqsubseteq \alpha$ and $k\gamma \sqsubseteq \beta$ by axiom R1. Since α and β are s-crisp, $\gamma \sqsubseteq \alpha$ and $\gamma \sqsubseteq \beta$. Thus $\gamma \sqsubseteq \alpha \sqcap \beta$ by axiom R1. □

In addition to be used in the definition of s-crisp relations, scalar relations also play an important rôle in other respects. Let us denote the set of all scalar relations by L. Then L is closed under the operations supremum $\sqcup$ and infimum $\sqcap$ by proposition 6.1.2(c) and axiom R1. So the tuple $(L, \sqsubseteq, \sqcap, \sqcup, O, \mathrm{id})$ is a complete distributive lattice, and it is a sublattice of the relation algebra $\mathcal{R}$ with the least element O and the greatest element id.

6.2 Strict Point Axiom

This section introduces a new concept of point relations and a strict point axiom. A concept of point relations was introduced by Schmidt and Ströhlein in [SS85, SS93] to give a simple proof of a representation theorem for Boolean relation algebras and apply such algebras to computer science. We made the concept more strict in section 3.3 to prove a representation theorem for fuzzy relation algebras. The concept of point relations is defined in this chapter in the spirit of section 3.3, but we have to attend to the difference between the notions of crispness in section 3.3 and in this section.

Before define the concept of point relations, we describe properties of relations which correspond to vectors in [SS85, SS93].

Proposition 6.2.1 *Let α be a s-crisp relation such that $\nabla\alpha = \alpha$. Then the following three conditions are equivalent:*

(a) $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$.

(b) $\nabla = \alpha\alpha^\sharp$.

(c) $\nabla = \alpha\nabla$.

Proof. (a)$\Longrightarrow$(b) If $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$, then $\nabla = \nabla\mathrm{id} \sqsubseteq \nabla\alpha\alpha^\sharp = \alpha\alpha^\sharp$.
(b)$\Longrightarrow$(c) If $\nabla = \alpha\alpha^\sharp$, then $\nabla = \alpha\alpha^\sharp \sqsubseteq \alpha\nabla$.
(c)$\Longrightarrow$(a) If $\nabla = \alpha\nabla$, then $\mathrm{id} = \mathrm{id} \sqcap \nabla = \mathrm{id} \sqcap \alpha\nabla \sqsubseteq \alpha(\alpha^\sharp\mathrm{id} \sqcap \nabla) = \alpha\alpha^\sharp$. □

The concept of point relations in relation algebras is defined as follows:

Definition 6.2.1 A **point relation** x is a s-crisp relation such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\mathrm{id} \sqsubseteq xx^\sharp$ and $\nabla x = x$. (Point relations will be denoted by lower case Roman letters such as $x, y, z, \cdots$.) The set of all point relations is denoted by X. □

Note that a point relation x is nonzero from its totality $\mathrm{id} \sqsubseteq xx^\sharp$. For point relations x and y, the relation $x^\sharp y$ is nonzero since $y \sqsubseteq x(x^\sharp y)$ by the totality $\mathrm{id} \sqsubseteq xx^\sharp$ of x.

Proposition 6.2.2 *Let x, x_0, y, y_0 be point relations and k a nonzero scalar. Then the following holds:*

(a) *If $kx \sqsubseteq y$, then $x = y$.*

(b) *If $kx^\sharp y \sqsubseteq x_0^\sharp y_0$, then $x = x_0$ and $y = y_0$.*

Proof. (a) Since y is s-crisp, it holds that $x \sqsubseteq y$. Using $\mathrm{id} \sqsubseteq xx^\sharp$, $x^\sharp \sqsubseteq y^\sharp$ and $y^\sharp y \sqsubseteq \mathrm{id}$ we have $y \sqsubseteq xx^\sharp y \sqsubseteq xy^\sharp y \sqsubseteq x$.
(b) Assume that $kx^\sharp y \sqsubseteq x_0^\sharp y_0$. Then we have $ky = k\nabla y = k\nabla x^\sharp y = \nabla kx^\sharp y \sqsubseteq \nabla x_0^\sharp y_0 = y_0$ by proposition 6.2.1 and so $y = y_0$ by (a). Similarly $x = x_0$ holds. □

By making use of our last definition of point relations in relation algebras, we add the following axiom:

Definition 6.2.2 A relation algebra $\mathcal{R}$ satisfies the **strict point axiom** iff:
R5. (a) For each nonzero relation α there are a nonzero scalar relation k and two point relations x and y such that $x\alpha y^\sharp = k\nabla$.
(b) $\sqcup_{x\in X} x^\sharp x = \mathrm{id}$. □

Note that condition (b) of the strict point axiom R5 is equivalent to $\sqcup_{x\in X} x = \nabla$. In what follows we assume that the fixed relation algebra $\mathcal{R}$ satisfies the strict point axiom R5.

Proposition 6.2.3 *Let α be a relation, x and y point relations. Then the following holds:*

(a) *If α is a nonzero relation, then there exist a nonzero scalar relation k and point relations x and y such that $kx^\sharp y \sqsubseteq \alpha$.*

(b) *If $x \neq y$, then $x \sqcap y = O$ and $xy^\sharp = O$.*

(c) *$x\alpha y^\sharp = k\nabla$ if and only if $\alpha \sqcap x^\sharp y = k(x^\sharp y)$.*

(d) *If $\alpha \sqsubseteq x^\sharp y$, then there exists a scalar relation k such that $\alpha = kx^\sharp y$.*

Proof. (a) If $\alpha \neq O$, then then there exist a nonzero scalar relation k and point relations x and y such that $x\alpha y^\sharp = k\nabla$ by the strict point axiom R5. Since x and y are point relations, it holds that

$$kx^\sharp y = kx^\sharp \nabla y = x^\sharp k \nabla y = x^\sharp x \alpha y^\sharp y \sqsubseteq \alpha$$

by proposition 6.1.2(b).
(b) Assume that $x \neq y$ and $x \sqcap y \neq O$. Then there exist a nonzero scalar relation k and point relations x_0 and y_0 such that $kx_0^\sharp y_0 \sqsubseteq x \sqcap y$ by (a). From proposition 6.2.2(e) $x \sqcap y$ is s-crisp, so it holds that $x_0^\sharp y_0 \sqsubseteq x \sqcap y$. Thus we have $y_0 = \nabla x_0^\sharp y_0 \sqsubseteq \nabla(x \sqcap y) \sqsubseteq \nabla x \sqcap \nabla y = x \sqcap y$ by proposition 6.2.1. Therefore $x = y_0 = y$ by axiom R1 and proposition 6.2.2(a). Finally, if $x \sqcap y = O$, then we have $xy^\sharp = xy^\sharp \sqcap \nabla \sqsubseteq (x \sqcap \nabla y)y^\sharp = (x \sqcap y)y^\sharp = O$.
(c) Assume that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Then it holds that

$$\begin{aligned} x\alpha y^\sharp &= x\alpha y^\sharp \sqcap \nabla \\ &= x\alpha y^\sharp \sqcap (xx^\sharp)(yy^\sharp) \\ &= x(\alpha \sqcap x^\sharp y)y^\sharp \\ &= x[k(x^\sharp y)]y^\sharp \\ &= k(xx^\sharp)(yy^\sharp) \\ &= k\nabla \end{aligned}$$

by propositions 6.2.1, 6.1.1(a) and 6.1.2(b). Next assume that $x\alpha y^\sharp = k\nabla$. Then we have

$$\alpha \sqcap x^\sharp y \sqsubseteq x^\sharp(x\alpha y^\sharp \sqcap \mathrm{id})y \sqsubseteq x^\sharp x \alpha y^\sharp y = x^\sharp(k\nabla)y = k(x^\sharp y)$$

by axiom R3, propositions 6.2.1 and 6.1.2(b). Conversely, it holds that

$$k(x^\sharp y) = k(x^\sharp \nabla y) = x^\sharp k \nabla y = x^\sharp(x\alpha y^\sharp)y \sqsubseteq \alpha$$

by proposition 6.1.2(b). Thus we have $k(x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y$.
(d) It is trivial that if $\alpha = O$ then $\alpha = O(x^\sharp y)$. Next assume that $\alpha \neq O$. Then, by the strict point axiom R5 and (c), there are a nonzero scalar relation k and point relations x_0, y_0 such that $\alpha \sqcap x_0^\sharp y_0 = k(x_0^\sharp y_0)$. Hence we have $k(x_0^\sharp y_0) \sqsubseteq \alpha \sqsubseteq x^\sharp y$, and so $x = x_0$ and $y = y_0$ by proposition 6.2.2(b), which implies $\alpha = k(x^\sharp y)$. □

By (d) of the last proposition, for every relation α and for every two point relations x, y there exists a scalar relation k such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, and so $x\alpha y^\sharp = k\nabla$ by (c) of the last proposition. Also, by proposition 6.1.2(d), such a scalar relation k is unique. For a relation α and point relations x, y, we define $\psi(\alpha)(x, y)$ to be the unique scalar relation k with $x\alpha y^\sharp = k\nabla$. Thus, by proposition 6.1.2(d), $\psi(\alpha)(x, y)$ is the unique scalar relation such that

$$x\alpha y^\sharp = \psi(\alpha)(x, y)\nabla \ .$$

Therefore $\psi(\alpha)$ defines an L-relation on the set X of all point relations in $\mathcal{R}$ since the set L of all scalar relations is a complete distributive lattice.

6.3 Representation Theorem

First we prove a representation theorem for relation algebras satisfying the strict point axiom R5. The representation problem of Boolean relation algebras was proposed by Tarski in [Tar41] and investigated for a long time, see [SS85, SS93, Mad91a] for more details on the history of the investigation of the representation theorem

for Boolean relation algebras. Also we proved an algebraic representation theorem of fuzzy relations in section 3.4, and proved such theorems for Dedekind categories (or allegories) and Zadeh categories in chapter 5. The following theorem also is a representation theorem for Dedekind categories with just one object.

Theorem 6.3.1 (Representation Theorem) *Let $\mathcal{R}$ be a relation algebra satisfying the strict point axiom. Then every relation α has a unique representation*

$$\alpha = \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x,y) y \ .$$

Proof. Since $\mathrm{id} = \sqcup_{x \in X} x^\sharp x$ and $\mathrm{id} = \sqcup_{y \in X} y^\sharp y$ by the strict point axiom R5, we have

$$\begin{aligned} \alpha &= \mathrm{id}\alpha\mathrm{id} \\ &= (\sqcup_{x \in X} x^\sharp x)\alpha(\sqcup_{y \in X} y^\sharp y) \\ &= \sqcup_{x,y \in X} x^\sharp x \alpha y^\sharp y \\ &= \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x,y) \nabla y \\ &= \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x,y) y \ . \end{aligned}$$

Finally we show the uniqueness of the representation. Assume that $\alpha = \sqcup_{x,y \in X} x^\sharp k_{x,y} y$. Then for all $x_0, y_0 \in X$ we have $\psi(\alpha)(x_0,y_0)\nabla = x_0 \alpha y_0^\sharp = \sqcup_{x,y \in X} x_0 x^\sharp k_{x,y} y y_0^\sharp = k_{x,y}$ by proposition 6.2.3(b). □

From the last theorem we can deduce the next property of the function $\psi : \mathcal{R} \to L\text{-}\mathbf{Rel}(X)$.

Corollary 6.3.1 *For every relation algebra $\mathcal{R}$ satisfying the strict point axiom, the function $\psi : \mathcal{R} \to L\text{-}\mathbf{Rel}(X)$ is bijective.*

Proof. If $\psi(\alpha) = \psi(\beta)$, then by the last theorem we have

$$\alpha = \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x,y)\nabla y = \sqcup_{x,y \in X} x^\sharp \psi(\beta)(x,y)\nabla y = \beta \ ,$$

which shows that ψ is injective. Given an L-relation $R \in L\text{-}\mathbf{Rel}(X)$, we set $\alpha_R = \sqcup_{x,y \in X} x^\sharp R(x,y)\nabla y$. Then by the uniqueness of the representation in the last theorem we have

$$R(x,y) = \psi(\alpha_R)(x,y) \ ,$$

which shows that ψ is surjective. □

The following proposition shows that $\psi : \mathcal{R} \to L\text{-}\mathbf{Rel}(X)$ preserves all operations of L-relations, that is, ψ is a homomorphism of relation algebras from $\mathcal{R}$ to $L\text{-}\mathbf{Rel}(X)$.

Proposition 6.3.1 *Let α, β be relations. Then the following holds:*

(a) $\psi(O) = 0_X$, $\psi(\nabla) = 1_X$ *and* $\psi(\mathrm{id}) = E_X$.

(b) *If $\alpha \sqsubseteq \beta$, then $\psi(\alpha) \sqsubseteq \psi(\beta)$.*

(c) $\psi(\alpha \sqcup \beta) = \psi(\alpha) \cup \psi(\beta)$,

(d) $\psi(\alpha \sqcap \beta) = \psi(\alpha) \cap \psi(\beta)$.

(e) $\psi(\alpha^\sharp) = \psi(\alpha)^\cup$.

(f) $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$.

Proof. (a) The first follows from $\psi(O)(x,y)\nabla = xOy^\sharp = O\nabla$, the second follows from $\psi(\nabla)(x,y)\nabla = x\nabla y^\sharp = \mathrm{id}\nabla$ by proposition 6.2.1. Noting $\psi(\mathrm{id})(x,y)\nabla = x\mathrm{id}y^\sharp = xy^\sharp$, the last follows from $\psi(\mathrm{id})(x,y)\nabla = \mathrm{id}\nabla$ if $x = y$ and $\psi(\mathrm{id})(x,y)\nabla = O\nabla$, otherwise by propositions 6.2.1 and 6.2.3(b).

(b) If $\alpha \sqsubseteq \beta$, then $\psi(\alpha)(x,y)\nabla = x\alpha y^\sharp \sqsubseteq x\beta y^\sharp = \psi(\beta)(x,y)\nabla$.
(c) follows from

$$\begin{aligned}\psi(\alpha \sqcup \beta)(x,y)\nabla &= x(\alpha \sqcup \beta)y^\sharp \\ &= x\alpha y^\sharp \sqcup x\beta y^\sharp \\ &= \psi(\alpha)(x,y)\nabla \sqcup \psi(\beta)(x,y)\nabla \\ &= [\psi(\alpha)(x,y) \sqcup \psi(\beta)(x,y)]\nabla \\ &= [\psi(\alpha) \cup \psi(\beta)](x,y)\nabla \ .\end{aligned}$$

(d) follows from

$$\begin{aligned}\psi(\alpha \sqcap \beta)(x,y)\nabla &= x(\alpha \sqcap \beta)y^\sharp \\ &= x\alpha y^\sharp \sqcap x\beta y^\sharp \\ &= \psi(\alpha)(x,y)\nabla \sqcap \psi(\beta)(x,y)\nabla \\ &= [\psi(\alpha)(x,y) \sqcap \psi(\beta)(x,y)]\nabla \\ &= [\psi(\alpha) \cap \psi(\beta)](x,y)\nabla \ ,\end{aligned}$$

by propositions 6.1.1(a) and 6.1.1(c) since x and y are point relations and $\psi(\alpha)(x,y), \psi(\beta)(x,y) \sqsubseteq \mathrm{id}$.
(e) follows from $\psi(\alpha^\sharp)(x,y)\nabla = x\alpha^\sharp y^\sharp = (y\alpha x^\sharp)^\sharp = (\psi(\alpha)(y,x)\nabla)^\sharp = \psi(\alpha)(y,x)\nabla = \psi(\alpha)^\cup(x,y)\nabla$, since $\psi(\alpha)(y,x)$ is a scalar relation.
(f) follows from

$$\begin{aligned}\psi(\alpha\beta)(x,y)\nabla &= x(\alpha\beta)y^\sharp \\ &= x\alpha\mathrm{id}\beta y^\sharp \\ &= x\alpha(\sqcup_{z\in X} z^\sharp z)\beta y^\sharp \\ &= \sqcup_{z\in X} x\alpha z^\sharp z\beta y^\sharp \\ &= \sqcup_{z\in X} \psi(\alpha)(x,z)\nabla\psi(\beta)(z,y)\nabla \\ &= \sqcup_{z\in X} \psi(\alpha)(x,z)\psi(\beta)(z,y)\nabla \\ &= \sqcup_{z\in X} [\psi(\alpha)(x,z) \sqcap \psi(\beta)(z,y)]\nabla \\ &= (\psi(\alpha)\psi(\beta))(x,y)\nabla \ ,\end{aligned}$$

since $\psi(\alpha)(x,z)$ and $\psi(\beta)(z,y)$ are scalar relations. □

It is now obvious that ψ^{-1} is a function and is a homomorphism of algebras of L-relations from L-**Rel**(X) to $\mathcal{R}$. Consequently the following corollary is deduced:

Corollary 6.3.2 (Isomorphism Theorem) *Every relation algebra $\mathcal{R}$ satisfying the strict point axiom is isomorphic to the algebra L-**Rel**(X) of L-relations on the set X of all point relations of $\mathcal{R}$, where L is the distributive lattice of scalar relations in $\mathcal{R}$.* □

In this chapter we proved a representation theorem for homogeneous relation algebras $\mathcal{R}$ satisfying the strict point axiom R5, which can be considered as Dedekind categories with just one object, using concepts of scalar relations and point relations. In section 5.1 such a theorem for Dedekind category was proved without using the concept of scalar relations. But in that section, the existence of the unit object was assumed to prove the theorem. The contribution of this chapter is so show that such a representation theorem can be proved without assuming the existence of a unit object, using instead our new algebraically defined concept of scalar relations.

Chapter 7

Crispness and Representation Theorem in Dedekind Categories

In this chapter we consider Dedekind categories named by Olivier and Serrato [OS95]. One of the aim of this chapter is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness was introduced in section 5.1 under the assumption that Dedekind categories have unit objects which are an abstraction of singleton (or one-point) sets. To capture the notion of crispness without such assumption, we use a notion of scalar relations. The notion of scalar relations in homogeneous relation algebras was introduced in section 6.1. The other aim of this chapter is to prove a representation theorem for Dedekind categories. Such a theorem for Dedekind categories with a unit object satisfying strict point axiom was also proved in section 5.1.

This chapter is organized as follows:

In section 7.1 we define a preorder among objects of Dedekind categories which compares the lattice structures on objects in a sense. Section 7.2 studies notions of scalars and crispness for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. In section 7.3 we recall the definition of L-relations, due to Goguen [Gog67], and illustrate a few relationships between crispness and lattice structures of scalars. In section 7.4 we show a representation theorem for uniform Dedekind categories satisfying the strict point axiom without the assumption of existence of unit objects, and it is proved that the representation function is a bijection preserving all operations of Dedekind categories.

7.1 Preorder among Objects of Dedekind Categories

In this section we provide a preorder among objects of Dedekind categories which compares the lattice structures on objects in a sense.

First, we define a function $\phi_W : \mathcal{D}(X,Y) \to \mathcal{D}(W,W)$ by

$$\phi_W(\xi) = \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W : W \rightharpoonup W$$

for a morphism $\xi : X \rightharpoonup Y$ and an object W of a Dedekind category $\mathcal{D}$. This function is related to scalars; the relationship will be described in the next section, and the following lemma holds:

Lemma 7.1.1 (a) $\phi_W(\xi)\nabla_{WZ} = \nabla_{WX}\xi\nabla_{YZ}$ *and* $\nabla_{ZW}\phi_W(\xi) = \nabla_{ZX}\xi\nabla_{YW}$ *for each object* Z.

(b) $\phi_W(\phi_X(\xi)) = \phi_W(\phi_Y(\xi)) = \phi_W(\xi)$.

(c) $\phi_W(\xi) = \phi_W(\xi^\sharp)$.

(d) *If* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $\xi \sqsubseteq \nabla_{XW}\phi_W(\xi)\nabla_{WY}$.

(e) *If* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $\phi_W(\xi) = O_{WW}$ *is equivalent to* $\xi = O_{XY}$.

Proof. (a) The former follows from

$$\begin{array}{rcl}
\phi_W(\xi)\nabla_{WZ} & = & (\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} \\
& \sqsubseteq & \nabla_{WX}\xi\nabla_{YW}\nabla_{WZ} \\
& \sqsubseteq & \nabla_{WX}\xi\nabla_{YZ} \\
& = & \nabla_{WX}\xi\nabla_{YZ} \sqcap \nabla_{WZ} \\
& \sqsubseteq & (\nabla_{WX}\xi\nabla_{YZ}\nabla^\sharp_{WZ} \sqcap \mathrm{id}_W)\nabla_{WZ} \\
& \sqsubseteq & (\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WZ} \\
& = & \phi_W(\xi)\nabla_{WZ} \ .
\end{array}$$

The latter is similar.

(b) follows from

$$\begin{array}{rcll}
\phi_W(\phi_X(\xi)) & = & \nabla_{WX}\phi_X(\xi)\nabla_{XW} \sqcap \mathrm{id}_W & (\text{ Definition of } \phi_W \ ) \\
& = & \nabla_{WX}\nabla_{XX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & (\ \phi_X(\xi)\nabla_{XW} = \nabla_{XX}\xi\nabla_{YW}\ ) \\
& = & \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & (\ \nabla_{WX}\nabla_{XX} = \nabla_{WX}\ ) \\
& = & \phi_W(\xi) & (\text{ Definition of } \phi_W \ )
\end{array}$$

and

$$\begin{array}{rcll}
\phi_W(\phi_Y(\xi)) & = & \nabla_{WY}\phi_Y(\xi)\nabla_{YW} \sqcap \mathrm{id}_W & (\text{ Definition of } \phi_W \ ) \\
& = & \nabla_{WX}\xi\nabla_{YY}\nabla_{YW} \sqcap \mathrm{id}_W & (\ \nabla_{WX}\phi_Y(\xi) = \nabla_{WX}\xi\nabla_{YY}\ ) \\
& = & \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & (\ \nabla_{YY}\nabla_{YW} = \nabla_{YW}\ ) \\
& = & \phi_W(\xi) & (\text{ Definition of } \phi_W \ ) \ .
\end{array}$$

$$\begin{array}{ccc}
\mathcal{D}(X,Y) & \xrightarrow{\phi_X} & \mathcal{D}(X,X) \\
{\scriptstyle \phi_Y}\downarrow & & \downarrow{\scriptstyle \phi_W} \\
\mathcal{D}(Y,Y) & \xrightarrow[\phi_W]{} & \mathcal{D}(W,W)
\end{array}$$

(c) follows from $\phi_W(\xi^\sharp) = (\phi_W(\xi^\sharp))^\sharp = (\nabla_{WY}\xi^\sharp\nabla_{XW} \sqcap \mathrm{id}_W)^\sharp = \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W = \phi_W(\xi)$.

(d) If $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, then

$$\begin{array}{rcl}
\xi & = & \xi \sqcap \nabla_{XY} \\
& = & \xi \sqcap \nabla_{XW}\nabla_{WY} \\
& \sqsubseteq & \nabla_{XW}(\nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W)\nabla_{WY} \\
& = & \nabla_{XW}\phi_{XYW}(\xi)\nabla_{WY} \ .
\end{array}$$

(e) is immediate from (d). □

A binary relation $\prec$ among objects of $\mathcal{D}$ is defined as follows: For two objects X and Y, the relation $X \prec Y$ holds if and only if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$. (Note that the three conditions $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$, $\mathrm{id}_X \sqsubseteq \nabla_{XY}\nabla_{YX}$ and $\phi_X(\mathrm{id}_Y) = \mathrm{id}_X$ are mutually equivalent.) It is easy to see that $\prec$ is a preorder, that is, reflexive and transitive. For $\nabla_{XX} = \nabla_{XX}\nabla_{XX}$, and if $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ and $\nabla_{YY} = \nabla_{YZ}\nabla_{ZY}$, then

$$\nabla_{XX} = \nabla_{XY}\nabla_{YY}\nabla_{YX} = \nabla_{XY}\nabla_{YZ}\nabla_{ZY}\nabla_{YX} \sqsubseteq \nabla_{XZ}\nabla_{ZX} \ .$$

Hence its symmetric kernel with $X \sim Y$ if and only if $X \prec Y$ and $Y \prec X$, is an equivalence relation. Note that in the category Rel_0 of example 2.3.1, two distinct objects are never equivalent.

Proposition 7.1.1 *Assume that $X \prec Y$. If $u \sqsubseteq \mathrm{id}_X$, $u \sqsubseteq \mathrm{id}_X$ and $u\nabla_{XY} \sqsubseteq v\nabla_{XY}$ for $u, v : X \rightharpoonup X$, then $u \sqsubseteq v$.*

Proof. It follows from $\nabla_{XX} = \nabla_{XY}\nabla_{YX}$ that $u = \mathrm{id}_X \sqcap u\nabla_{XX} = \mathrm{id}_X \sqcap u\nabla_{XY}\nabla_{YX}$. □

Definition 7.1.1 A Dedekind category $\mathcal{D}$ is **uniform** if all pairs of objects of $\mathcal{D}$ are equivalent, that is, if $X \sim Y$ for all objects X and Y of $\mathcal{D}$. □

A morphism $f : X \rightharpoonup Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) and $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function* and may be introduced as $f : X \to Y$.

Proposition 7.1.2 (a) *If there exists at least one total morphism $\alpha : X \rightharpoonup Y$, then $X \prec Y$.*

(b) *If there exists at least one function* $f : X \to Y$, *then* $X \prec Y$.

(c) *If* $X \prec W$ *or* $Y \prec W$, *then* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$.

(d) *If* $X \prec Y$ *and* $\nabla_{XY} = \nabla_{XW}\nabla_{WY}$, *then* $X \prec W$.

(e) *If* $\nabla_{XY} = p^\sharp q$ *for some functions* $p : W \to X$ *and* $q : W \to Y$ *and if* $X \prec Y$, *then* $X \sim W$.

Proof. (a) Assume that α is total, then we have $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp \sqsubseteq \nabla_{XY}\nabla_{YX}$.
(b) is a just corollary of (a).
(c) If $\nabla_{XX} = \nabla_{XW}\nabla_{WX}$, then $\nabla_{XY} = \nabla_{XX}\nabla_{XY} = \nabla_{XW}\nabla_{WX}\nabla_{XY} \sqsubseteq \nabla_{XW}\nabla_{WY}$.
(d) $\nabla_{XX} = \nabla_{XY}\nabla_{YX} = \nabla_{XW}\nabla_{WY}\nabla_{YX} \sqsubseteq \nabla_{XW}\nabla_{WX}$.
(e) First note that $W \prec X$ by (a). Next $\nabla_{XY} = p^\sharp q \sqsubseteq \nabla_{XW}\nabla_{WX}$ and so it follows from (d) that $X \prec W$. □

7.2 Scalars and Crispness

We now introduce the two notions of scalars and of s-crisp relations as a preparation for defining a concept of points with a separation property, that is, different points never meet.

Definition 7.2.1 A **scalar** k on X is a morphism $k : X \multimap X$ of $\mathcal{D}$ such that $k \sqsubseteq \mathrm{id}_X$ and $k\nabla_{XX} = \nabla_{XX}k$.□

A scalar k on X commutes with all endomorphisms $\alpha : X \multimap X$, that is, $k\alpha = \alpha k$, because

$$k\alpha = \alpha \sqcap k\nabla_{XX} = \alpha \sqcap \nabla_{XX}k = \alpha k \ .$$

It is trivial that the zero morphism $O_{XX} : X \multimap X$ and the identity morphism $\mathrm{id}_X : X \multimap X$ are scalars on X. The set of all scalars on X is denoted by $\mathcal{F}(X)$. It is clear that $\mathcal{F}(X)$ is a complete distributive lattice for all objects X. A morphism $\xi : X \multimap Y$ is called an *ideal* if $\nabla_{XX}\xi\nabla_{YY} = \xi$. The notion of ideals in relation algebras was initially introduced by Jónsson and Tarski [JT52], see section 4.1. The following lemma shows that scalars bijectively correspond to ideals.

Lemma 7.2.1 (a) *If* $\iota : X \multimap X$ *is an ideal, then* $k = \iota \sqcap \mathrm{id}_X$ *is a scalar on* X *such that* $\iota = k\nabla_{XX}$.

(b) *If* k *is a scalar on* X, *then* $\iota = k\nabla_{XX}$ *is an ideal such that* $k = \iota \sqcap \mathrm{id}_X$.

Proof. (a) Assume that ι is an ideal on an object X, then we have

$$(\iota \sqcap \mathrm{id}_X)\nabla_{XX} \sqsubseteq \iota\nabla_{XX} = \iota = \iota \sqcap \mathrm{id}_X\nabla_{XX} \sqsubseteq (\iota\nabla_{XX}^\sharp \sqcap \mathrm{id}_X)\nabla_{XX} = (\iota \sqcap \mathrm{id}_X)\nabla_{XX} \ ,$$

and so $(\iota \sqcap \mathrm{id}_X)\nabla_{XX} = \iota = \nabla_{XX}(\iota \sqcap \mathrm{id}_X)$.
(b) Assume that k is a scalar on an object X, then we have

$$\nabla_{XX}(k\nabla_{XX})\nabla_{XX} = k\nabla_{XX}\nabla_{XX}\nabla_{XX} = k\nabla_{XX}$$

and

$$k = k\mathrm{id}_X = k = k\nabla_{XX} \sqcap \mathrm{id}_X \ . \qquad \square$$

Proposition 7.2.1 *Let* $\xi : X \multimap Y$ *be a morphism. Then the following holds:*

(a) $\phi_W(\xi)$ *is a scalar on* W.

(b) *If* $X \prec Y$, *then* $\phi_X(\phi_Y(k)) = k$ *for all scalars* $k \in \mathcal{F}(X)$.

(c) *If* $X \sim Y$, *then* $\mathcal{F}(X)$ *and* $\mathcal{F}(Y)$ *are isomorphic as lattices.*

(d) $\phi_X(k)\xi = \xi\phi_Y(k)$ *for all scalars* k *on* W.

(e) *If* $\xi \neq O_{XY}$, *then there is a nonzero scalar* $k \in \mathcal{F}(X)$ *such that* $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$.

Proof. (a) Set $W = Z$ in Lemma 7.1.1(a). Then $\phi_W(\xi)\nabla_{WW} = \nabla_{WX}\xi\nabla_{YW} = \nabla_{WW}\phi_W(\xi)$.
(b) First note that $\phi_Y(k)\nabla_{YX} = \nabla_{YX}k\nabla_{XX}$ by Lemma 7.1.1(a) and so

$$\begin{array}{rcll} \nabla_{XY}\phi_Y(k)\nabla_{YX} & = & \nabla_{XY}\nabla_{YX}k\nabla_{XX} & \\ & = & \nabla_{XX}k\nabla_{XX} & (\text{by } \nabla_{XX} = \nabla_{XY}\nabla_{YX}\ ) \\ & = & k\nabla_{XX} & (\text{since } k \text{ is a scalar}\ )\ . \end{array}$$

Hence we have

$$\begin{array}{rcl} \phi_X(\phi_Y(k)) & = & \nabla_{XY}\phi_Y(k)\nabla_{YX} \sqcap \mathrm{id}_X \\ & = & k\nabla_{XX} \sqcap \mathrm{id}_X \\ & = & k\ . \end{array}$$

(c) is obvious from (b).
(d) By Lemma 7.1.1(a) we have $\phi_X(k)\nabla_{XY} = \nabla_{XW}k\nabla_{WY} = \nabla_{XY}\phi_Y(k)$ and consequently $\phi_X(k)\alpha = \alpha \sqcap \phi_X(k)\nabla_{XY} = \alpha \sqcap \nabla_{XY}\phi_Y(k) = \alpha\phi_Y(k)$.
(e) Set $k = \phi_X(\xi)$. Then it is clear that k is a scalar on X by (a) and $\nabla_{XX}\xi\nabla_{YY} = k\nabla_{XY}$ by Lemma 7.1.1(a). And k is nonzero by Lemma 7.1.1(d), since ξ is nonzero. (Cf. [KFM96, Theorem 5.4]) □

From the above Lemma 7.1.1(a) we have ϕ_W as a mapping $\phi_W : \mathcal{D}(X,Y) \to \mathcal{F}(W)$.

Fact 7.2.1

$$\begin{array}{rcll} \phi_W(\phi_X(\xi)) & = & \nabla_{WX}\phi_X(\xi)\nabla_{XW} \sqcap \mathrm{id}_W & (\text{Definition of } \phi_W) \\ & = & \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & (\phi_X(\xi)\nabla_{XW} = \nabla_{XX}\xi\nabla_{YW}) \\ & = & \nabla_{WX}\xi\nabla_{YX}\nabla_{XW} \sqcap \mathrm{id}_W & (\nabla_{WX}\phi_X(\xi) = \nabla_{WX}\xi\nabla_{YX}) \end{array}$$

and

$$\begin{array}{rcll} \phi_W(\phi_Y(\xi)) & = & \nabla_{WY}\phi_Y(\xi)\nabla_{YW} \sqcap \mathrm{id}_W & (\text{Definition of } \phi_W) \\ & = & \nabla_{WX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & (\nabla_{WY}\phi_Y(\xi) = \nabla_{WX}\xi\nabla_{YY}) \\ & = & \nabla_{WY}\nabla_{YX}\xi\nabla_{YW} \sqcap \mathrm{id}_W & (\phi_Y(\xi)\nabla_{YX} = \nabla_{YX}\xi\nabla_{YX})\ . \end{array}$$ □

In particular, the following holds for $\xi = \nabla_{XY}$:

$$\begin{array}{rcl} \nabla_{WX}\nabla_{XY}\nabla_{YW} \sqcap \mathrm{id}_W & = & \nabla_{WX}\nabla_{XY}\nabla_{YX}\nabla_{XW} \sqcap \mathrm{id}_W \\ & = & \nabla_{WY}\nabla_{YX}\nabla_{XY}\nabla_{YW} \sqcap \mathrm{id}_W\ . \end{array}$$

Jónsson and Tarski [JT52] suggested that a Boolean relation algebra which satisfies Tarski rule has no ideal except for the zero relation and the universal relation. The next proposition corresponds to the suggestion.

Proposition 7.2.2 *If the original Tarski rule holds in* $\mathcal{D}$, *that is, all nonzero morphisms* $\alpha : X \to X$ *satisfy* $\nabla_{XX}\alpha\nabla_{XX} = \nabla_{XX}$, *then there is no scalar on* X *except for the zero morphism* O_{XX} *and the identity* id_X.

Proof. Let k be a nonzero scalar on X. Then, by the Tarski rule, we have

$$k\nabla_{XX} = k\nabla_{XX}\nabla_{XX} = \nabla_{XX}k\nabla_{XX} = \nabla_{XX}\ ,$$

which means that k is total, and so $\mathrm{id}_X \sqsubseteq kk^\sharp = k$ by $k \sqsubseteq \mathrm{id}_X$. □

By using the notion of scalar, we define a crispness which called s-crispness (scalar crispness).

Definition 7.2.2 A morphism $\alpha : X \to Y$ is **s-crisp** if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all nonzero scalars $k : X \to X$ and all morphisms $\tau : X \to Y$. □

It is trivial from the above definition that every universal morphism ∇_{XY} is s-crisp.

Proposition 7.2.3 (a) *A morphism is s-crisp if and only if its converse is s-crisp.*

(b) *The infimum of two s-crisp morphisms is s-crisp.*

(c) *If $f : X \to Y$ is a function and a morphism $\beta : Y \rightharpoonup Z$ is s-crisp, then the composite $f\beta : X \rightharpoonup Z$ is s-crisp.*

(d) *If the identity id_Y is s-crisp, then so are all functions $f : X \to Y$.*

(e) *A morphism $\alpha : X \rightharpoonup Y$ is s-crisp if and only if its relative pseudo-complement $\alpha' \Rightarrow \alpha$ is s-crisp for every morphism $\alpha' : X \rightharpoonup Y$.*

Proof. (a) Assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha^\sharp$ for a nonzero scalar k on Y and a morphism $\tau : Y \rightharpoonup X$. Then $\phi_X(k)\tau^\sharp = \tau^\sharp k = (k\tau)^\sharp \sqsubseteq (\alpha^\sharp)^\sharp = \alpha$ and so $\tau^\sharp \sqsubseteq \alpha$, since $\phi_X(k)$ is a nonzero scalar on X by Lemma 7.1.1(e). Hence $\tau \sqsubseteq \alpha^\sharp$.
(b) Assume that $\alpha_i : X \rightharpoonup Y$ is s-crisp for $i = 0$ or 1 and $k\tau \sqsubseteq \alpha_0 \sqcap \alpha_1$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. Then we have $k\tau \sqsubseteq \alpha_0$ and $k\tau \sqsubseteq \alpha_1$, and so $\tau \sqsubseteq \alpha_0$ and $\tau \sqsubseteq \alpha_1$ by s-crispness. Hence $\tau \sqsubseteq \alpha_0 \sqcap \alpha_1$.
(c) Assume that $k\tau \sqsubseteq f\beta$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Z$. First note that $\phi_Y(k)$ is a nonzero scalar by Lemma 7.1.1(e) and $\phi_Y(k)f^\sharp = f^\sharp k$ by proposition 7.2.1(d). Then we have

$$\phi_Y(k)f^\sharp\tau = f^\sharp k\tau \sqsubseteq f^\sharp f\beta \sqsubseteq \beta$$

and so $f^\sharp\tau \sqsubseteq \beta$ by the s-crispness of β. Therefore $\tau \sqsubseteq ff^\sharp\tau \sqsubseteq f\beta$, which completes the proof.
(d) is a special case of (b).
(e) First assume that $\alpha : X \rightharpoonup Y$ is s-crisp and $k\tau \sqsubseteq \alpha' \Rightarrow \alpha$ for a nonzero scalar k and morphisms $\tau, \alpha' : X \rightharpoonup Y$. Then we have $k(\tau \sqcap \alpha') = k\tau \sqcap \alpha' \sqsubseteq \alpha$ and so $\tau \sqcap \alpha' \sqsubseteq \alpha$, since $\alpha : X \rightharpoonup Y$ is s-crisp. Therefore $\tau \sqsubseteq \alpha' \Rightarrow \alpha$. Conversely, if $\alpha' \Rightarrow \alpha$ is s-crisp for all morphisms $\alpha' : X \rightharpoonup Y$, then $\alpha = \nabla_{XY} \Rightarrow \alpha$ is s-crisp. This completes the proof. □

It immediately follows from the last proposition 7.2.3(c) that every composite of s-crisp functions is also an s-crisp function.

A morphism $\alpha : X \rightharpoonup Y$ is *complemented* if it has a complement morphism $\overline{\alpha} : X \rightharpoonup Y$ such that $\alpha \sqcup \overline{\alpha} = \nabla_{XY}$ and $\alpha \sqcap \overline{\alpha} = O_{XY}$.

Theorem 7.2.1 *The following four statements are equivalent:*

(a) *If $k \neq O_{XX}$ and $k \sqcap k' = O_{XX}$ for scalars $k, k' \in \mathcal{F}(X)$, then $k' = O_{XX}$.*

(b) *The zero morphism O_{XY} is s-crisp for every object Y (that is, if $k\tau = O_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$, then $\tau = O_{XY}$).*

(c) *For every morphism $\alpha : X \rightharpoonup Y$, its pseudo-complement $\neg\alpha : X \rightharpoonup Y$ is s-crisp.*

(d) *Every complemented morphism $\alpha : X \rightharpoonup Y$ is s-crisp.*

Proof. (a)$\Longrightarrow$(b) Assume that $k\tau = O_{XY}$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup Y$. Recall that $\phi_X(\tau)$ is a scalar on X. Hence we have

$$\begin{aligned} k \sqcap \phi_X(\tau) = k\phi_X(\tau) &= k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \\ &\sqsubseteq k\nabla_{XX}\tau\nabla_{XY} \\ &= \nabla_{XX}k\tau\nabla_{YX} \\ &= O_{XX} \ . \end{aligned}$$

It follows from (a) that $\phi_X(\tau) = O_{XX}$ and so $\tau = O_{XY}$ by Lemma 7.1.1(e). Hence O_{XY} is s-crisp.
(b)$\Longrightarrow$(a) is trivial.
(b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) is a corollary of the last lemma. □

Definition 7.2.3 A scalar k on X is called **linear** if and only if for every scalar k' on X an equation $k \sqcap k' = O_{XX}$ implies $k' = O_{XX}$. □

Let $\mathcal{W}(X)$ denote the set of all linear scalars on X. Every identity id_X is obviously linear. Note that a scalar k on X is linear if and only if its pseudo-complement $\neg k$ $(= \mathrm{id}_X \sqcap (k \Rightarrow O_{XX}))$ in $\mathcal{F}(X)$ is equal to O_{XX}.

Lemma 7.2.2 *If X is a nonempty object, then $\mathcal{W}(X)$ is a filter of $\mathcal{F}(X)$.*

Proof. 0) It is trivial that O_{XX} is not a linear scalar, whenever X is nonempty.
i) If $k_0, k_1 \in \mathcal{W}(X)$, then $k_0 \sqcap k_1 \in \mathcal{W}(X)$: Assume $(k_0 \sqcap k_1) \sqcap k' = O_{XX}$. Then $k_0 \sqcap (k_1 \sqcap k') = O_{XX}$ and so $k_1 \sqcap k' = O_{XX}$, which shows $k' = O_{XX}$.
ii) If $k_0 \in \mathcal{W}(X)$ and $k_1 \in \mathcal{F}(X)$ with $k_0 \sqsubseteq k_1$, then $k_1 \in \mathcal{W}(X)$: Assume $k_1 \sqcap k' = O_{XX}$. Then $k_0 \sqcap k' = O_{XX}$ and so $k' = O_{XX}$. □

So the set of linear scalars on X is a sublattice of the lattice $\mathcal{F}(X)$ of all scalars on X, and as such it is distributive.

Definition 7.2.4 A morphism $\alpha : X \rightharpoonup Y$ is **l-crisp** if $k\tau \sqsubseteq \alpha$ implies $\tau \sqsubseteq \alpha$ for all linear scalars $k : X \rightharpoonup X$ and all morphisms $\tau : X \rightharpoonup Y$. □

Proposition 7.2.4 *Every zero morphism O_{XY} is l-crisp.*

Proof. Assume that $k\tau = O_{XY}$ for a linear scalar on X and a morphism $\tau : X \rightharpoonup Y$. Then we have

$$\begin{array}{rcl} k \sqcap \phi_X(\tau) & = & k\phi_X(\tau) \\ & = & k(\nabla_{XX}\tau\nabla_{YX} \sqcap \mathrm{id}_X) \\ & \sqsubseteq & k\nabla_{XX}\tau\nabla_{YX} \\ & \sqsubseteq & \nabla_{XX}k\tau\nabla_{YX} \\ & = & O_{XY} \end{array}$$

and so $\phi_X(\tau) = O_{XX}$. Hence $\tau = O_{XY}$ by Lemma 7.1.1(e). □

7.3 Crispness in L-Relations

Obviously an L-relation $k : X \rightharpoonup X$ is a scalar on X if and only if

$$\forall x, x' \in X : k(x,x) = k(x',x') \text{ and } x \neq x' \Longrightarrow k(x,x') = \mathbf{0} \ .$$

An L-relation $R : X \rightharpoonup Y$ is called *0-1* crisp [Gog67] if $R(x,y) = \mathbf{0}$ or $R(x,y) = \mathbf{1}$ for all $(x,y) \in X \times Y$. Of course the zero relation 0_{XY}, the universal relation 1_{XY} and the identity relation E_X are 0-1 crisp. For a 0-1 crisp L-relation $R : X \rightharpoonup Y$ define an L-relation $\overline{R} : X \rightharpoonup Y$ by $\overline{R}(x,y) = \mathbf{0}$ if $R(x,y) = \mathbf{1}$ and $\overline{R}(x,y) = \mathbf{1}$ otherwise. Then $R \cup \overline{R} = 1_{XY}$ and $R \cap \overline{R} = 0_{XY}$. This fact means that all 0-1 crisp L-relations are complemented.

Proposition 7.3.1 *All s-crisp L-relations are 0-1 crisp.*

Proof. Let an L-relation $R : X \rightharpoonup Y$ be s-crisp. Assume that $a = R(x_0, y_0)$ is not equal to $\mathbf{0} \in L$ for some point $(x_0, y_0) \in X \times Y$. Consider a scalar k on X such that $k(x,x') = a$ if $x = x'$ and $k(x,x') = \mathbf{0}$ otherwise, and an L-relation $T : X \rightharpoonup Y$ such that $T(x,y) = a \Rightarrow R(x,y)$ for all $(x,y) \in X \times Y$. Then we have $kT \sqsubseteq R$, since

$$(kT)(x,y) = a \wedge (a \Rightarrow R(x,y)) \leq R(x,y)$$

for all $(x,y) \in X \times Y$. Hence $T \sqsubseteq R$ follows from the fact that $R : X \rightharpoonup Y$ is s-crisp. Finally we have $\mathbf{1} = (a \Rightarrow a) = T(x_0, y_0) \leq R(x_0, y_0)$, which shows R is 0-1 crisp. □

The converse of the last proposition does not hold in general. Its necessary and sufficient condition is given by the following:

Proposition 7.3.2 *For L-relations the following statements are equivalent:*

C0. $\forall a, b \in L : a \wedge b = \mathbf{0} \Longrightarrow a = \mathbf{0}$ *or* $b = \mathbf{0}$.

K0. All 0-1 crisp L-relations are s-crisp.

Proof. First assume that C0 and $kT \sqsubseteq R$ for a scalar k on X, an L-relation $T : X \rightharpoondown Y$, and a 0-1 crisp L-relation $R : X \rightharpoondown Y$. To prove that R is s-crisp we have to show that $T(x, y) \leq R(x, y)$ for all $(x, y) \in X \times Y$. Since $R(x, y) = \mathbf{0}$ or $\mathbf{1}$ by the 0-1 crispness of R it is enough to show that if $R(x, y) = \mathbf{0}$ then $T(x, y) = \mathbf{0}$. But $(kT)(x, y) = k(x, x) \wedge T(x, y) \leq R(x, y)$. Hence when $R(x, y) = \mathbf{0}$, we have $T(x, y) = \mathbf{0}$ from C0 and $k(x, x) \neq \mathbf{0}$. Conversely assume that K0 and $a \wedge b = \mathbf{0}$ for $a, b \in L$. Define a scalar k on a singleton set $I = \{*\}$ and an L-relation $R : I \to I$ by $k(*, *) = a$ and $T(*, *) = b$, respectively. Then $kT = 0_{II}$ and so $k = 0_{II}$ or $T = 0_{II}$ since 0_{II} is s-crisp by the assumption K0. □

Proposition 7.3.3 *For L-relations the following statements are equivalent:*

C1. $\forall a, b \in L : a \wedge b = \mathbf{0}$ *and* $a \vee b = \mathbf{1} \Longrightarrow a = \mathbf{0}$ *or* $b = \mathbf{0}$.

K1. All complemented L-relations are 0-1 crisp.

K2. All L-relations which are functions are 0-1 crisp.

Proof. Trivial. □

Definition 7.3.1 An element x of a lattice L is called **linear** if $x \wedge y = \mathbf{0}$ implies $y = \mathbf{0}$ for $y \in L$. □

Let $k : X \rightharpoondown X$ be an L-relation on a nonempty set X. If k is a linear scalar, then $k(x, x)$ is linear in L for all $x \in X$.

Assume that $k(x, x) \wedge a = \mathbf{0}$ for $a \in L$. Now consider a scalar $k' : X \rightharpoondown X$ such that $k'(x, x') = a$ if $x = y$, and $k'(x, x') = \mathbf{0}$ otherwise. Then $k \cap k' = 0_{XX}$ and so $k' = 0_{XX}$ by the linearity of k. Hence $a = \mathbf{0}$, which proves that $k(x, x)$ is linear.

Proposition 7.3.4 *All 0-1 crisp L-relations are l-crisp.*

Proof. Let an L-relation $R : X \rightharpoondown Y$ be 0-1 crisp and assume that $kT \sqsubseteq R$ for a linear scalar k on X and an L-relation $T : X \rightharpoondown Y$. We have to show that $T(x, y) \leq R(x, y)$ for all $(x, y) \in X \times Y$. Now $k(x, x) \wedge T(x, y) \leq R(x, y) = (kT)(x, y) \subseteq R(x, y)$, and since $k(x, x)$ is linear, it follows that $R(x, y) = \mathbf{0}$ implies $T(x, y) = \mathbf{0}$, which is sufficient since $R(x, y)$ can only be $\mathbf{0}$ or $\mathbf{1}$ by 0-1 crispness. □

The converse of the above proposition does not hold: Consider a Boolean lattice L having a nontrivial element s such that $s \neq \mathbf{0}$ and $s \neq \mathbf{1}$, and define an L-relation $R_s : X \rightharpoondown X$ by $R(x, x') = s$ if $x = x'$ and $R(x, x') = \mathbf{0}$ otherwise. Then it is clear that R_s is l-crisp, but not 0-1 crisp. Generally for a Boolean lattice L every L-relation is l-crisp since the identity E_X is a unique linear scalar on X.

7.4 Representation Theorem

In this section we first introduce the concept of points in Dedekind categories. Then some useful properties on points, due to Schmidt and Ströhlein [SS85], and a point axiom will be stated to show a representation theorem in uniform Dedekind categories. In particular, the point axiom induces a function assigning a concrete L-relation between the sets of point relations to an abstract relation in Dedekind categories. In view of [Fur97b, KF95, SS85] the concept of points in Dedekind categories is defined as follows:

Definition 7.4.1 Let $\mathcal{D}$ be a Dedekind category. A **point** x of X is an s-crisp function $x : X \to X$ such that $\nabla_{XX} x = x$. □

We will denote the set of all points of X by $\chi(X)$.

Lemma 7.4.1 *Let x and x' be points of X. Then the following holds:*

(a) *If $\nabla_{XX}\rho = \rho$ and $\rho \sqsubseteq x$ for a morphism $\rho : X \to X$, then $\rho = kx$ for a unique scalar k on X.*

(b) *If $x \neq x'$, then $x \sqcap x' = O_{XX}$ and $xx'^\sharp = O_{XX}$.*

Proof. (a) First set $k = \phi_X(\rho x^\sharp)$. Then by proposition 7.2.1(a) k is a scalar on X, and $k = \rho x^\sharp \sqcap \mathrm{id}_X$ from $\nabla_{XX}x = x$ and $\nabla_{XX}\rho = \rho$. Moreover we have $\rho = \rho \sqcap x \sqsubseteq (\rho x^\sharp \sqcap \mathrm{id}_X)x \sqsubseteq \rho x^\sharp x \sqsubseteq \rho$. Finally the uniqueness of k follows from $k = k\nabla_{XX} \sqcap \mathrm{id}_X = kx\nabla_{XX} \sqcap \mathrm{id}_X = \rho\nabla_{XX} \sqcap \mathrm{id}_X$.
(b) It is enough to show that if $x \sqcap x' \neq O_{XX}$ then $x = x'$. As $x \sqcap x' \sqsubseteq x$ and $\nabla_{XX}(x \sqcap x') = x \sqcap x'$, by (a) there is a unique scalar $k : X \to X$ such that $x \sqcap x' = kx$. If $x \sqcap x' \neq O_{XX}$, then $k \neq O_{XX}$ and so $x \sqsubseteq x'$, because $kx \sqsubseteq x'$ and x' is s-crisp. If $x \sqcap x' = O_{XX}$, then $xx'^\sharp = xx'^\sharp \sqcap \nabla_{XX} \sqsubseteq (x \sqcap \nabla_{XX}x')x'^\sharp = (x \sqcap x')x'^\sharp = O_{XX}$. This completes the proof. □

Set $L = \mathcal{F}(W)$ for a fixed object W. Then L is a complete distributive lattice. A function $\chi(\alpha) : \chi(X) \times \chi(Y) \to L$ assigning $\chi(\alpha)(x, y) = \phi_W(x\alpha y^\sharp) \in L$ to a pair (x, y) of points x of X and y of Y, gives an L-relation of $\chi(X)$ into $\chi(Y)$. Thus we have a function $\chi : \mathcal{D}(X, Y) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(Y))$.

Proposition 7.4.1 *If $\mathcal{D}$ is a uniform Dedekind category, then the function $\chi : \mathcal{D}(X, Y) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(Y))$ satisfies the following properties:*

(a) *$\chi(O_{XY}) = 0_{\chi(X)\chi(Y)}$, $\chi(\nabla_{XY}) = 1_{\chi(X)\chi(Y)}$ and $\chi(\mathrm{id}_X) = E_{\chi(X)}$.*

(b) *$\chi(\alpha \sqcup \alpha') = \chi(\alpha) \cup \chi(\alpha')$ and $\chi(\alpha \sqcap \alpha') = \chi(\alpha) \cap \chi(\alpha')$.*

(c) *$\chi(\alpha^\sharp) = \chi(\alpha)^\cup$.*

(d) *$\chi(\alpha)\chi(\beta) = \chi(\alpha[\bigsqcup_{y \in \chi(Y)} y^\sharp y]\beta)$.*

(e) *The function $\chi : \mathcal{D}(X, Y) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(Y))$ is surjective.*

Proof. Recall that $\chi(\alpha)(x, y) = \phi_W\phi_X(x\alpha y^\sharp) = \phi_W\phi_Y(x\alpha y^\sharp)$ by Lemma 7.1.1(b).
(a) It is immediate that $\chi(O_{XY})(x, y) = O_{WW}$. Note that $x\nabla_{XY}y^\sharp = \nabla_{XY}$ from $x\nabla_{XX} = \nabla_{XX}$ and $y\nabla_{YY} = \nabla_{YY}$. The second equality follows from

$$\begin{array}{rcll} \phi_W(x\nabla_{XY}y^\sharp) & = & \phi_W(\nabla_{XY}) & (\text{by } x\nabla_{XX} = \nabla_{XX} \text{ and } y\nabla_{YY} = \nabla_{YY}) \\ & = & \phi_W\phi_X(\nabla_{XY}) & (\text{by Lemma 7.1.1(b)}) \\ & = & \phi_W(\mathrm{id}_X) & (\text{by } X \sim Y) \\ & = & \mathrm{id}_W & (\text{by } X \sim W) \end{array}$$

and the third holds from $\phi_X(x\mathrm{id}_X x'^\sharp) = \nabla_{XX}xx'^\sharp\nabla_{XX} \sqcap \mathrm{id}_X = xx'^\sharp \sqcap \mathrm{id}_X$ and Lemma 7.4.1(b).
(b) The former equality is trivial from $\phi_W(x(\alpha \sqcup \alpha')y^\sharp) = \phi_W(x\alpha y^\sharp) \sqcup \phi_W(x\alpha' y^\sharp)$, and the latter follows from

$$\begin{array}{rcl} \phi_W(x(\alpha \sqcap \alpha')y^\sharp) & = & \nabla_{WX}(x\alpha y^\sharp \sqcap x\alpha' y^\sharp)\nabla_{YW} \sqcap \mathrm{id}_W \\ & \sqsubseteq & \nabla_{WX}x\alpha y^\sharp\nabla_{YW} \sqcap \nabla_{WX}x\alpha' y^\sharp\nabla_{YW} \sqcap \mathrm{id}_W \\ & (= & \phi_W(x\alpha y^\sharp) \sqcap \phi_W(x\alpha' y^\sharp)\ ) \\ & \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap \nabla_{XW}\nabla_{WX}x\alpha' y^\sharp\nabla_{YW}\nabla_{WY})\nabla_{YW} \sqcap \mathrm{id}_W \\ & \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap \nabla_{XX}x\alpha' y^\sharp\nabla_{YY})\nabla_{YW} \sqcap \mathrm{id}_W \\ & \sqsubseteq & \nabla_{WX}(x\alpha y^\sharp \sqcap x\alpha' y^\sharp)\nabla_{YW} \sqcap \mathrm{id}_W \\ & = & \phi_W(x(\alpha \sqcap \alpha')y^\sharp)\ . \end{array}$$

(c) It directly follows from Lemma 7.1.1(c).

(d) First note that $\chi(\alpha)(x,y) \sqcap \chi(\beta)(y,z) = \chi(\alpha y^\sharp y \beta)(x,z)$ for $(x,y,z) \in \chi(X) \times \chi(Y) \times \chi(Z)$, since

$$\begin{array}{rcl}
\phi_W(x\alpha y^\sharp) \sqcap \phi_W(y\beta z^\sharp) & = & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp (\nabla_{YW} \sqcap y\alpha^\sharp x^\sharp \nabla_{XW} \nabla_{WY} y\beta z^\sharp \nabla_{ZW}) \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& = & \nabla_{WX} x\alpha y^\sharp y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& (= & \phi_W(x\alpha y^\sharp y\beta z^\sharp)\ ) \\
& = & (\nabla_{WX} x\alpha y^\sharp \sqcap \nabla_{WZ} z\beta^\sharp y^\sharp)(y\alpha^\sharp x^\sharp \nabla_{XW} \sqcap y\beta z^\sharp \nabla_{ZW}) \sqcap \mathrm{id}_W \\
& \sqsubseteq & \nabla_{WX} x\alpha y^\sharp \nabla_{YW} \sqcap \nabla_{WY} y\beta z^\sharp \nabla_{ZW} \sqcap \mathrm{id}_W \\
& = & \phi_W(x\alpha y^\sharp) \sqcap \phi_W(y\beta z^\sharp)\ .
\end{array}$$

Therefore we have

$$\begin{array}{rcl}
\chi(\alpha)\chi(\beta)(x,z) & = & \sqcup_{y\in\chi(Y)}[\chi(\alpha)(x,y) \sqcap \chi(\beta)(y,z)] \\
& = & \sqcup_{y\in\chi(Y)}\chi(\alpha y^\sharp y\beta)(x,z) \\
& = & \chi(\alpha[\sqcup_{y\in\chi(Y)} y^\sharp y]\beta)(x,z)\ .
\end{array}$$

(e) Let $R : \chi(X) \rightharpoonup \chi(Y)$ be an L-relation. Noticing $L = \mathcal{F}(W)$ we define a morphism $\alpha_R : X \rightharpoonup Y$ by

$$\alpha_R = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(R(x,y)) x^\sharp \nabla_{XY} y\ .$$

Then we have $\phi_X(x_0\alpha_R y_0^\sharp) = \phi_X(R(x_0,y_0))$ from $\phi_X(x_0\alpha_R y_0^\sharp)\nabla_{XY} = x_0\alpha_R y_0{}^\sharp = \phi_X(R(x_0,y_0))\nabla_{XY}$. Hence we have

$$\chi(\alpha_R)(x_0,y_0) = \phi_W(x_0\alpha_R y_0^\sharp) = \phi_W\phi_X(x_0\alpha_R y_0^\sharp) = \phi_W\phi_X(R(x_0,y_0)) = R(x_0,y_0)\ ,$$

which completes the proof. □

Definition 7.4.2 A Dedekind category $\mathcal{D}$ satisfies the **strict point axiom** iff:
D5. $\sqcup_{x\in\chi(X)} x = \nabla_{XX}$ for all objects X. □

Assume that $\sqcup_{x\in\chi(X)} x = \nabla_{XX}$. Then it follows from $\mathrm{id}_X \sqcap x \sqsubseteq (\mathrm{id}_X x^\sharp \sqcap \mathrm{id}_X)x \sqsubseteq x^\sharp x$ that

$$\begin{array}{rcl}
\mathrm{id}_X & = & \mathrm{id}_X \sqcap \nabla_{XX} \\
& = & \mathrm{id}_X \sqcap (\sqcup_{x\in\chi(X)} x) \\
& = & \sqcup_{x\in\chi(X)} (\mathrm{id}_X \sqcap x) \\
& \sqsubseteq & \sqcup_{x\in\chi(X)} x^\sharp x\ .
\end{array}$$

Hence $\sqcup_{x\in\chi(X)} x^\sharp x = \mathrm{id}_X$. Conversely assume that $\sqcup_{x\in\chi(X)} x^\sharp x = \mathrm{id}_X$. Then

$$\begin{array}{rcl}
\nabla_{XX} & = & \nabla_{XX}\mathrm{id}_X \\
& = & \nabla_{XX}(\sqcup_{x\in\chi(X)} x^\sharp x) \\
& = & \sqcup_{x\in\chi(X)} \nabla_{XX} x^\sharp x \\
& = & \sqcup_{x\in\chi(X)} \nabla_{XX} x \\
& = & \sqcup_{x\in\chi(X)} x\ .
\end{array}$$

Therefore the condition $\sqcup_{x\in\chi(X)} x = \nabla_{XX}$ is equivalent to $\sqcup_{x\in\chi(X)} x^\sharp x = \mathrm{id}_X$.

Proposition 7.4.2 *If a Dedekind category $\mathcal{D}$ satisfies the strict point axiom, then for all objects X the identity morphism id_X is complemented. Moreover, if the statement (a) of theorem 7.2.1 is valid in $\mathcal{D}$, then id_X is s-crisp.*

Proof. Assume that $\nabla_{XX} = \sqcup_{x\in\chi(X)} x$. Then it is obvious that

$$\nabla_{XX} = \nabla_{XX}\nabla_{XX} = (\sqcup_{x\in\chi(X)} x^\sharp)(\sqcup_{y\in\chi(X)} y) = \mathrm{id}_X \sqcup (\sqcup_{x\neq y\in\chi(X)} x^\sharp y)\ .$$

Here note that for $x \neq y \in \chi(X)$ we have $\mathrm{id}_X \sqcap x^\sharp y \sqsubseteq x^\sharp(x\mathrm{id}_X \sqcap y) = O_{XX}$. Hence this shows that $\sqcup_{x\neq y\in\chi(X)} x^\sharp y$ is the complement of id_X. □

Theorem 7.4.1 (Representation Theorem) *Assume that $\mathcal{D}$ is a uniform Dedekind and satisfies the strict point axiom. Then every morphism $\alpha : X \to Y$ has a unique representation*

$$\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y \ ,$$

where $k_{x,y}$ is a scalar on X for all $(x, y) \in \chi(X) \times \chi(Y)$.

Proof. Note that $x\alpha y^\sharp = \phi(x\alpha y^\sharp)\nabla_{XY}$ for $x \in \chi(X)$ and $y \in \chi(Y)$, because $x\alpha y^\sharp = \nabla_{XX} x\alpha y^\sharp \nabla_{YY} = \phi_X(x\alpha y^\sharp)\nabla_{XY}$ by Lemma 7.1.1(a). We now show the uniqueness of the representation. Assume

$$\alpha = \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} k_{x,y} x^\sharp \nabla_{XY} y \ .$$

Then for all $(x, y) \in \chi(X) \times \chi(Y)$ we have $k_{x,y}\nabla_{XY} = x\alpha y^\sharp = \phi_X(x\alpha y^\sharp)\nabla_{XY}$ and so $k_{x,y} = \phi_X(x\alpha y^\sharp)$ by proposition 7.1.1. Hence it suffices to see that $\alpha = \sqcup_{x\in\chi(X)}\sqcup_{y\in\chi(Y)}\phi_X(x\alpha y^\sharp)x^\sharp\nabla_{XY}y$. Since $\mathrm{id}_X = \sqcup_{x\in\chi(X)}x^\sharp x$ and $\mathrm{id}_Y = \sqcup_{y\in\chi(Y)}y^\sharp y$ by the strict point axiom D5, we have

$$\begin{aligned}
\alpha &= \mathrm{id}_X \alpha \mathrm{id}_Y \\
&= (\sqcup_{x\in\chi(X)} x^\sharp x)\alpha(\sqcup_{y\in\chi(Y)} y^\sharp y) \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp x\alpha y^\sharp y \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} x^\sharp \phi_X(x\alpha y^\sharp)\nabla_{XY} y \\
&= \sqcup_{x\in\chi(X)} \sqcup_{y\in\chi(Y)} \phi_X(x\alpha y^\sharp) x^\sharp \nabla_{XY} y \ .
\end{aligned}$$

This completes the proof. □

Corollary 7.4.1 *A uniform Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if the function $\chi : \mathcal{D}(X, X) \to L\text{-}\mathbf{Rel}(\chi(X), \chi(X))$ is injective for all objects X.*

Proof. First assume that the function χ is injective. Then it follows from proposition 7.4.1(a) and (d) that $\mathrm{id}_X = \sqcup_{x\in\chi(X)} x^\sharp x$, which is equivalent to $\nabla_X = \sqcup_{x\in\chi(X)} x$. Secondly assume that the point axiom and consequently the representation theorem 7.4.1 hold. Let $\chi(\alpha) = \chi(\alpha')$ for $\alpha, \alpha' : X \to Y$. Then $\phi_W(x\alpha y^\sharp) = \phi_W(x\alpha' y^\sharp)$ for all $(x, y) \in \chi(X) \times \chi(Y)$. Since $\mathcal{D}$ is uniform, $\phi_X(x\alpha y^\sharp) = \phi_Y(x\alpha' y^\sharp)$ for all $(x, y) \in \chi(X) \times \chi(Y)$and so $\alpha = \alpha'$ by the virtue of the representation theorem. □

From the proof of proposition 2.3.1(d) it is easy to see that $\nabla_{XY} \neq O_{XY}$ for all nonempty objects X and Y if $\mathcal{D}$ has a unit object I and satisfies the strict point axiom D5.

As a result we have proved that a Dedekind category which has a unit object satisfying the strict point axiom is equivalent to a subcategory of a category of L-relations.

Let I and X be objects in $\mathcal{D}$. An I-point of X is an s-crisp function $p : I \to X$ such that $p = \nabla_{II} p$. Thus, when I is a unit object in $\mathcal{D}$, an I-point of X is just an s-crisp function from I to X. The set of all I-points of X will be denoted by $Q(X)$.

Proposition 7.4.3 *Let I and X be objects in $\mathcal{D}$. Then the following holds:*

(a) *If $X \prec I$, then a morphism $x = \nabla_{XI} p : X \to X$ is a point of X for an I-point $p : I \to X$ of X.*

(b) *If $I \prec X$, then a morphism $p = \nabla_{IX} x : I \to X$ is an I-point of X for a point $x : X \to X$ of X.*

(c) *If $X \sim I$, then $\nabla_{IX} = \sqcup_{p\in Q(X)} p$ is equivalent to $\nabla_{XX} = \sqcup_{x\in\chi(X)} x$.*

Proof. (a) First note that

$$\nabla_{XX} x = \nabla_{XX}\nabla_{XI} p = \nabla_{XI} p = x \ ,$$

$$x^\sharp x = (\nabla_{XI} p)^\sharp (\nabla_{XI} p) = p^\sharp \nabla_{IX} \nabla_{XI} p \sqsubseteq p^\sharp \nabla_{II} p = p^\sharp p \sqsubseteq \mathrm{id}_X \ ,$$

and

$$x x^\sharp = (\nabla_{XI} p)(\nabla_{XI} p)^\sharp = \nabla_{XI} p p^\sharp \nabla_{IX} \sqsupseteq \nabla_{XI}\nabla_{IX} = \nabla_{XX}$$

by $X \prec I$. Next assume that $k\tau \sqsubseteq \nabla_{XI}p(=x)$ for a nonzero scalar k on X and a morphism $\tau : X \rightharpoonup X$. Then $\phi_I(k)\nabla_{IX}\tau = \nabla_{IX}k\tau \sqsubseteq \nabla_{IX}\nabla_{XI}p \sqsubseteq \nabla_{II}p = p$ and so $\nabla_{IX}\tau \sqsubseteq p$, since $\phi_I(k) \neq O_{II}$ by Lemma 7.1.1(e) and p is s-crisp. Hence $\tau \sqsubseteq \nabla_{XX}\tau = \nabla_{XI}\nabla_{IX}\tau \sqsubseteq \nabla_{XI}p = x$ by $X \prec I$.
(b) First note that

$$\nabla_{II}p = \nabla_{II}\nabla_{IX}x = \nabla_{IX}x = p \ ,$$

$$p^\sharp p = (\nabla_{IX}x)^\sharp(\nabla_{IX}x) = x^\sharp\nabla_{XI}\nabla_{IX}x \sqsubseteq x^\sharp\nabla_{XX}x = x^\sharp x \sqsubseteq \mathrm{id}_X \ ,$$

and

$$pp^\sharp = (\nabla_{IX}x)(\nabla_{IX}x)^\sharp = \nabla_{IX}xx^\sharp\nabla_{XI} = \nabla_{IX}\nabla_{XX}\nabla_{XI} = \nabla_{II} \sqsupseteq \mathrm{id}_I$$

by $I \prec X$. Next assume that $k\tau \sqsubseteq \nabla_{IX}x(=p)$ for a nonzero scalar k on I and a morphism $\tau : I \rightharpoonup X$. Then $\phi_X(k)\nabla_{XI}\tau = \nabla_{XI}k\tau \sqsubseteq \nabla_{XI}\nabla_{IX}x \sqsubseteq \nabla_{XX}x = x$ and so $\nabla_{XI}\tau \sqsubseteq x$, since $\phi_X(k) \neq O_{XX}$ by Lemma 7.1.1(e) and x is s-crisp. Hence $\tau \sqsubseteq \nabla_{II}\tau = \nabla_{IX}\nabla_{XI}\tau \sqsubseteq \nabla_{IX}x = p$ by $I \prec X$.
(c) First assume that $\nabla_{IX} = \sqcup_{p\in Q(X)}p$. Then

$$\sqcup_{x\in\chi(X)}x = \sqcup_{p\in Q(X)}\nabla_{XI}p = \nabla_{XI}\sqcup_{p\in Q(X)}p = \nabla_{XI}\nabla_{IX} = \nabla_{XX}$$

by $X \prec I$. Conversely assume that $\nabla_{XX} = \sqcup_{x\in\chi(X)}x$. Then we have

$$\sqcup_{p\in Q(X)}p = \sqcup_{x\in\chi(X)}\nabla_{IX}x = \nabla_{IX}\sqcup_{x\in\chi(X)}x = \nabla_{IX}\nabla_{XX} = \nabla_{IX} \ . \qquad \square$$

In this chapter, we defined a notion of s-crisp and points. Unfortunately s-crispness is not equivalent to 0-1 crispness in L-relations but just a sufficient condition for 0-1 crispness. So we gave a condition the two crispness to be equivalent. However the notion of s-crispness is enough to make points satisfy separate property, and we proved representation theorem for Dedekind categories without assumption of existence of unit objects.

Chapter 8

Conclusion

The contribution of this thesis is as follows:

(1) We proposed two new algebraic formalisations of fuzzy relations which are fuzzy relation algebras and Zadeh categories, and proved their representation theorems. To prove such theorems, we used a notion of point relations with a separation property, that is, different point relations never meet. In order to make point relations satisfy this property, a notion of crispness is necessary. In both formalisations, we defined notions of crispness via scalar multiplications, which is equivalent to an intuitive element-wise definition of crispness of fuzzy relations, namely 0-1 crispness.

(2) We proved representation theorems for relation algebras and Dedekind categories. As in (1) of fuzzy relations, we used a notion of point relations. Since neither relation algebras nor Dedekind categories have scalar multiplications, we introduced a notion of scalar relations and defined the crispness by using this notion of scalar relations. Of course the crispness also provided the separation property of point relations.

We see the following opportunities for further research:

(a) As we described in (1), the notion of crispness in fuzzy relation algebras and Zadeh categories is well defined via scalar multiplication. But in relation algebras and Dedekind categories, the notion is not so well defined, that is, crispness in these two frameworks is not equivalent to 0-1 crispness of L-relations. The notion of s-crispness is just a sufficient condition for 0-1 crispness of L-relations.

(b) We would like to investigate the aspect of new applications of our fuzzy relational calculus.

The suggestion (a) proposes the necessity to continue studying crispness in Dedekind categories. The author is especially interested in the case that L-relations take values in a Boolean algebra; for example in the power set $\mathcal{P}(\{a, b\})$ of a set $\{a, b\}$. In this case, the notion of s-crispness is too strict to characterize 0-1 crispness in Dedekind categories.

In spite of (b), already, fuzzy relation algebras [KF95] which were introduced in chapter 3 gave a theoretical basis to theory of fuzzy difunctional dependency in fuzzy relational databases [OJ96], and a rewriting system of fuzzy graphs by using single pushouts [MoK97] based on a study of Zadeh categories [KFM96] which were introduced in chapter 5. Besides that, in the future, our calculus would be applied to *graded accessibility* and *fuzzy possible world semantics* introduced by Suzuki [Suz96]. In the research, accessibility relations correspond to L-relations which satisfy condition C0 provided in chapter 7. These L-relations may be useful tool to investigate accessibility and reliability in networks. Also the results in applying the Boolean relation algebraic approach to the theory of natural languages [Bot92a, Bot92b, Sup76, Sup79, Sup81] suggest that our calculus may enable them to treat fuzziness in an element-free style. Furthermore, in order to consider applications of our calculus to relational modelling of fuzzy systems, we should study fuzzy relational equations in our frameworks.

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